SOLAR ROTATION FROM 2D INVERSION

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ABSTRACT

A 2-D regularized least square inversion code for solar rotation has been constructed which approximates the rotation rate by piecewise polynomials projected on B-splines. It is applied to the rotational splitting data of BBSO (Ref. 1). A discussion of the influence of the number and order of the spline basis on the results is given. Preliminary results of inversion with LOI data (Ref. 2) are presented.

Keywords: inverse problem, solar rotation.

1. BASIC EQUATIONS

The rotational splitting $\Delta\nu_{nlm}$ of a p-mode of degree l, radial order n and frequency $\nu_{n,l,m}$ can be expressed as an integral over the rotation rate with weighting kernels depending on the non dimensioned radius and on the colatitude θ according to:

$$\Delta\nu_{nlm} = \int_0^1 \int_{-1}^1 K_{nl}(r) P_l^m(\mu)^2 \Omega(r,\mu) dr d\mu \qquad [1]$$

where $\mu = cos(\theta)$, $\Omega(r,\mu)$ is the unknown rotation rate. The splitting $\Delta\nu_{nlm}$ is defined by

$$\Delta\nu_{nlm} = \frac{\nu_{n,l,m} - \nu_{n,l,m=0}}{m}$$

The kernels $K_{nl}(r)$ depend on the model and on the radial eigen function of the mode n,l (Ref. 3).

Due to the symmetry of the weighting kernels relatively to μ, only the symmetric component of the rotation rate relatively to the equatorial plane can be obtained. We search for the unknown rotation rate as a linear combination of piecewise functions $\varphi_p(r)$ and $\psi_q(\mu^2)$.

$$\Omega(r,\mu) = \sum_{p=1}^{N_r^\Omega} \sum_{q=1}^{N_\mu^\Omega} \omega_{pq}\ \varphi_p(r)\psi_q(\mu^2) \qquad [2]$$

We introduce the following vectors and matrix:

$$W \equiv (W_k)_{k=1..N} \qquad k \equiv (n,l,m) \qquad W_k = \Delta\nu_{nlm}$$

$$\overline{\Omega} \equiv (\overline{\Omega}_Q)_{Q=1..N_r^\Omega N_\mu^\Omega} \qquad Q \equiv (p,q) \qquad \overline{\Omega}_Q = \omega_{pq}$$

$$R \equiv (R_{kQ})_{\substack{k=1..N \\ Q=1..N_{r\mu}^\Omega}} \qquad \text{with} \quad N_{r\mu}^\Omega = N_r^\Omega N_\mu^\Omega \quad \text{and}$$

$$R_{k,Q} = 2\left[\int_0^1 K_{nl}(r)\varphi_p(r)dr\right]\left[\int_0^1 P_l^m(\mu)^2 \psi_q(\mu^2) d\mu\right]$$

Thus the inversion problem writes:

$$W \ = \ R\,\overline{\Omega} \qquad [3]$$

The observations are not yet accurate enough to give all the splittings $\Delta\nu_{nlm}$ of a mode n,l. The observational splittings are given by the coefficients $a_i^{n,l}$ of their development on $P_i(m/L)$ functions, with their errors $\delta a_i^{n,l}$. The inversions can be performed either on the $a_i^{n,l}$ or on the splittings. Here we have derived the splittings and their standard deviations σ_{nlm} according to the following expressions:

$$\Delta\nu_{nlm} = \frac{L}{m}\sum_j a_j^{n,l} P_j\left(\frac{m}{L}\right) \qquad [4]$$

with $\quad L = \sqrt{l(l+1)}$

$$\sigma_{nlm} = \sqrt{\left(\frac{L}{m}\right)^2 \sum_j (\delta a_j^{n,l})^2 P_j\left(\frac{m}{L}\right)^2} \qquad [5]$$

In this study we assume that the errors on the splittings are independent. This may introduce some errors on the solution since this assumption is not valid for the multiplets of a given mode (l,n). A regularization term $T_{r\mu}$ is also introduced to avoid the large spurious variations of the solution induced by the ill-conditioned inversion problem. We thus minimize the quantity:

$$J_{r\mu}(\overline{\Omega}) = \|PW - PR\overline{\Omega}\|^2 + T_{r\mu} \qquad [6]$$

$$P \equiv \left(\frac{1}{\sigma_k}\delta(k,j)\right)_{\substack{k=1..N \\ j=1..N}}$$

$$T_{r\mu} = \int_0^1 \int_0^1 \left[\lambda_r f_r \left(\frac{\partial^2\Omega}{\partial r^2}\right)^2 + \lambda_\mu f_\mu \left(\frac{\partial^2\Omega}{\partial(\mu^2)^2}\right)^2\right] dr d\mu$$

$$T_{r\mu} = \overline{\Omega}^\top [\lambda_r Z^r + \lambda_\mu Z^\mu]\,\overline{\Omega} = \overline{\Omega}^\top Z \overline{\Omega}$$

with :

$$Z \equiv (z_{QQ_1})_{\substack{Q=1..N^\Omega \\ Q_1=1..N^\Omega}} \qquad Q \equiv (p,q) \qquad Q_1 \equiv (p_1,q_1)$$

$$z_{QQ_1} = \lambda_r \left[\int_0^1 \varphi_p''\ \varphi_{p_1}''\ dr\right]\left[\int_0^1 \psi_q\ \psi_{q_1}\ d\mu\right] + \lambda_\mu \left[\int_0^1 \varphi_p\ \varphi_{p_1}\ dr\right]\left[\int_0^1 \psi_q''\ \psi_{q_1}''\ d\mu\right]$$

The estimation of the rotation rate at a point r_0 and μ_0 is derived from the vector solution $\overline{\Omega}$ of the minimization of equation [6]:

$$\tilde{\Omega}(r_0, \mu_0) = \xi^\top \overline{\Omega} \qquad [7]$$

$$\xi \equiv (\xi_Q)_{Q=1..N^{\Omega}_{r\mu}} \qquad \xi_Q = \varphi_{p(Q)}(r_0)\ \psi_{q(Q)}(\mu_0^2)$$

From the propagation of the errors we derive the standard deviation on the rotation rate at that point by the relation:

$$\sigma^2(\tilde{\Omega}(r_0, \mu_0)) = \xi^\top B_{\overline{\Omega}}\ \xi \qquad [8]$$

$$= \sum_{Q=1}^{N^{\Omega}} \sum_{Q_1=1}^{N^{\Omega}} \xi_Q\ (B_{\overline{\Omega}})_{QQ_1}\ \xi_{Q_1}$$

where the covariance matrix $B_{\overline{\Omega}}$ on the solution $\overline{\Omega}$ is given by

$$B_{\overline{\Omega}} = (C)^{-1} R^\top P^2 R (C)^{-1} \qquad C = R^\top P^2 R + Z$$

More details on the derivation of the equations can be found in Ref. 4. The inversion depends on the numbers N^{Ω}_r and N^{Ω}_μ of the piecewise polynomials $\varphi(r)$ and $\psi(\mu^2)$, of the order of the spline functions and of the distribution of the fitting points of these polynomials. In what follows, the distribution of these points along the radius has been chosen according to the density of the turning points of the considered p-modes set of data. The inversion depends also on the values and forms of the regularizing term through the coefficients λ_r, λ_μ and the functions f_r and f_μ.

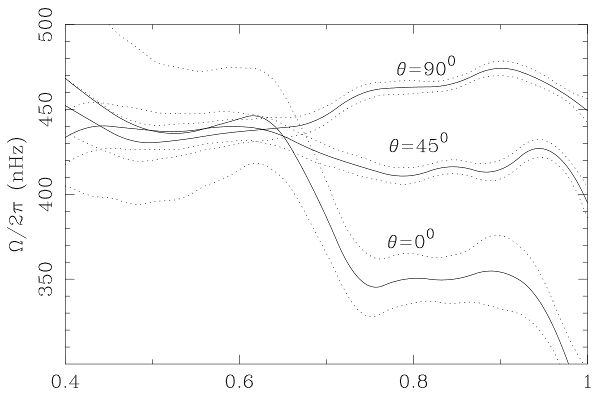

Figure 1: Variation of the rotation rate Ω as a function of the radius at three latitudes: polar, mid and equatorial latitudes. Dotted lines represent $\Omega \pm \sigma$.

2. INVERSION OF BBSO DATA

2.1 Results for the solar rotation rate

We have applied our code to derive the internal solar rotation rate from the 1986 Libbrecht data [1] (Ref. 1), hereafter referenced as BBSO, for modes l=5 to 60. We use the a_i coefficients and their errors to reconstruct the splittings and their errors according to equation [4] and [5]. The resulting solar rotation rate is shown in Figure 1 as a function of the radius for three latitudes: 0, 45, 90 degree. It has been obtained using $N^{\Omega}_r = 20$ and $N^{\Omega}_\mu = 10$ piecewise polynomials projected on cubic splines basis and with regularizing parameters: $\lambda_r = 10^{-6}$ $\lambda_\mu = 10^{-6}$ $f_r = f_\mu = 1$

Dotted curves represent the 1σ errors on the solution $(\Omega \pm \sigma)$. The solution is not valid for $r > 0.85$ due to the lack of modes with degrees $l > 60$. It has no significance too for radius lower than 0.4. Our results are in agreement with previously published results (Ref. 5,6,7,8). The rotation rate has a surface like latitudinal dependence in the whole convection zone and depends only on the radius in the internal radiative zone with a rapid variation at the basis of the convection zone.

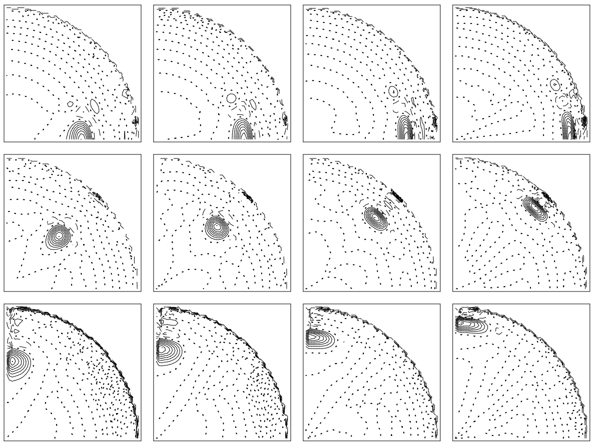

Figure 2: The contour plots of the averaging kernels $\overline{K}(r_0, \mu_0, r, \mu)$ are given for three values of the latitude $\mu_0 = 0.01, 0.707, 0.99$ and four values of the radius $r_0 = 0.55, 0.65, 0.75, 0.85$. Dotted lines correspond to zero values contour plots and dashed lines to negative values contour plots.

2.2 Averaging kernels

The estimation of the rotation at a point r_0, μ_0 given by [7] can be expressed as a linear combination of the data $W_k = \Delta\nu_{nlm}$:

$$\tilde{\Omega}(r_0, \mu_0) = \sum_{k=1}^{N} c_k(r_0, \mu_0) W_k$$

$$= \int_0^1 dr \int_{-1}^1 d\mu \overline{K}(r_0, \mu_0, r, \mu) \Omega(r, \mu)$$

with $\overline{K}(r_0, \mu_0, r, \mu) = \sum_{k=1}^{N} c_k(r_0, \mu_0)\ K_k(r, \mu)$.

The estimated rotation rate at the point r_0 and μ_0 appears to be the average of the real rotation rate weighted by an averaging kernel $\overline{K}(r_0, \mu_0, r, \mu)$. The

[1] p-mode data acquired by Ken Libbrecht and Martin Woodard, Big Bear Solar Observatory, Caltech

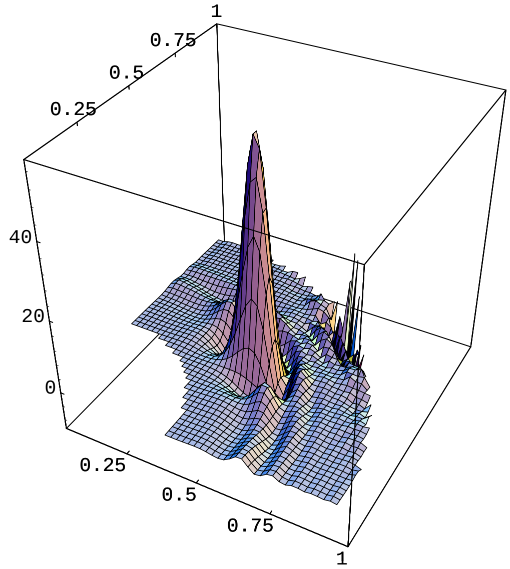

Figure 3: Two dimensional averaging kernel at mid-latitude for r_0= 0.55. It is well peaked close to the location r_0, μ_0 (μ_0 =0.707). We see some contribution from the solar surface due to the lack of modes of large degrees.

solution will be well spatially resolved if the averaging kernels are close to a δ distribution in radius and μ around the target radius r_0 and target μ_0.

Contour plots of averaging kernels for different target radius r_0 and μ_0 are given in Figure 2 and a three dimensional plot of an averaging kernel ($r_0 = 0.55, \mu_0 = 0.707$) is shown in Figure 3. It is seen that the kernels are peaked around the target values but with some contributions at the surface.

The spatial resolution at a point (r_0, μ_0) can be estimated from the characteristics of the averaging kernels. It depends principally on the set of modes we consider and on the regularizing parameters which are used to smooth the solution. It increases with lower regularizing term, at the expense of larger errors on the solution. Different ways of estimating the spatial resolution can be used (Ref. 9).

Here we characterize the averaging kernels by the contour plot curve C corresponding to half of the height of the maximum value (Fig. 4) and by different quantities. Δ is the geometrical distance between position (r, θ) of the maximum value of the averaging kernel and the target point (r_0, θ_0). The radial and latitudinal half width of the curve C, Δr and $r\Delta\theta$, are used to define the spatial resolution of the inversion. These quantities are multiplied by the standard deviation $\sigma(r_0, \mu_0)$ given by equation [8] which represents the error on the rotation rate $\Omega(r_0, \mu_0)$ given by propagation of the errors. We thus take into account the opposite variation of errors on the solution and its spatial resolution in order to give an estimate the quality of the result of the inversion. All these quantities are given in Figure 4 for $r = 0.55$. It is seen that, as already discussed (see for example Ref. 9), the quality of the inversion is much better

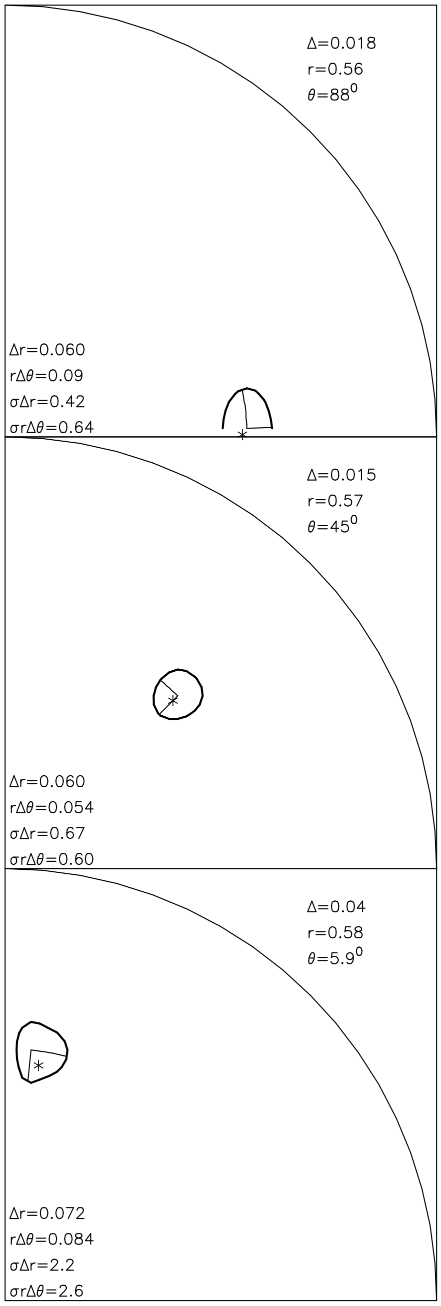

Figure 4: Contour plot curve C of the averaging kernels for the target radius r_0=0.55 (indicated by a * point) and three latitudes (equator, mid-latitude and pole) corresponding to half height of the maximum value.

for equator and mid-latitude rotation rate than at the pole.

The variation of $\sigma\Delta r$ and $\sigma r\Delta\theta$ relatively to the number of piecewise polynomials is plotted in Figure 5. The results show that these quantities do not vary very significantly and that 20 piecewise polynomials are enough to describe the rotation rate. We have also found that these quantities are not much sensitive to the order of the B-splines that we used.

3. PRELIMINARY RESULTS ON INVERSIONS WITH LOI DATA.

We have considered the LOI splittings given for the degrees $l = 2, 3, 4, 5$ by Appourchaux et al. (Ref. 2) and added them to the BBSO data for $5 < l \leq 60$. The inversion has been made for two values of the regularizing parameters $\lambda_r = 10^{-5}$ $\lambda_\mu = 5.10^{-5}$ (solution 1) and $\lambda_r = 10^{-6}$ $\lambda_\mu = 5.10^{-6}$ (solution 2) and with $f_r = 1$ $f_\mu = r^{-4}$ (Ref. 9). The results are given for the three latitudes $\mu_0 = 0.01, 0.707, 0.99$ for the BBSO data only on the left and for the BBSO+LOI data on the right side in Figure 6. We see that in the two cases the modification of the regularizing constants induces a larger difference in the solution behavior for radius smaller than 0.4, contrarily to what happens for $r > 0.4$.

The contour plots given in Figure 2 for the averaging kernels appear to be the same for the two sets of data. Adding the LOI data does not improve significantly the spatial resolution in our computations. The results obtained with the two values of the regularizing parameters show as expected, that the spatial resolution estimated by the quantities Δr and $r\Delta\theta$ is better for lower values of λ_r and λ_μ but the errors on the solution (Fig. 6) are larger. The positions of the maximum of the averaging kernels are also closer to the respective points r_0, μ_0 in that case. However it appears that the product of the error σ by the latitudinal and radial resolution $\sigma\Delta r$ and $\sigma r\Delta\theta$ are lower for the larger values of the regularizing constants. As an example, all these quantities are reported in Table I for a radius r_0= 0.35 and for the three latitudes: equator (E), mid-latitude (M), pole (P). From these results, we are lead to favour solution 1 (Fig. 6 upper right panel) which gives an estimated solar rotation rate almost independent of the latitude for radius $0.2 < r < 0.6$.

4. CONCLUSION

We have developed a 2D least square inversion code for the solar rotation rate, using piecewise polynomials projected on B-spline functions and we have derived some quantities to test the quality of the solution. The study of the sensitivity of the results relatively to the inversion parameters gives an optimal number of piecewise polynomials to be considered

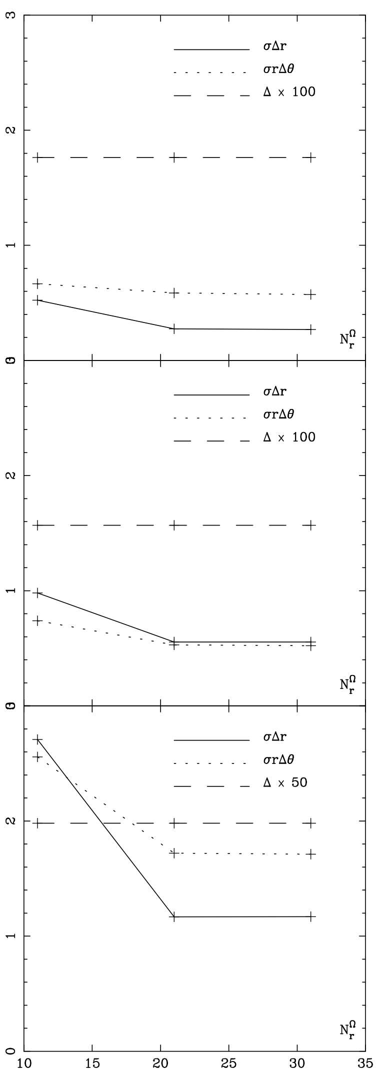

Figure 5: Sensitivity of the radial and latitudinal spatial resolution multiplied by the error on the solution $\sigma\Delta r$ and $\sigma r\Delta\theta$ relatively to the number of piecewise polynomials N_r^Ω. They are given for three values $N_r^\Omega = 10, 20, 30$ at the same location ($r_0 = 0.55$) and ($\mu_0 = 0.01, 0.707, 0.99$) than Figure 4 The position of the maximum value of the averaging kernel, characterized by Δ, is not modified within the grid of r and μ we use to compute these kernels.

Table 1:

	λ_r	Δ	Δr	$r\Delta\theta$	$\sigma\Delta r$	$\sigma r\Delta\theta$
E	10^{-5}	0.13	0.08	0.11	0.72	0.97
	10^{-6}	0.10	0.076	0.10	1.07	1.4
M	10^{-5}	0.1	0.09	0.14	1.08	1.73
	10^{-6}	0.06	0.08	0.13	1.62	2.41
P	10^{-5}	0.21	0.1	0.22	2.	4.4
	10^{-6}	0.15	0.09	0.17	3.3	5.8

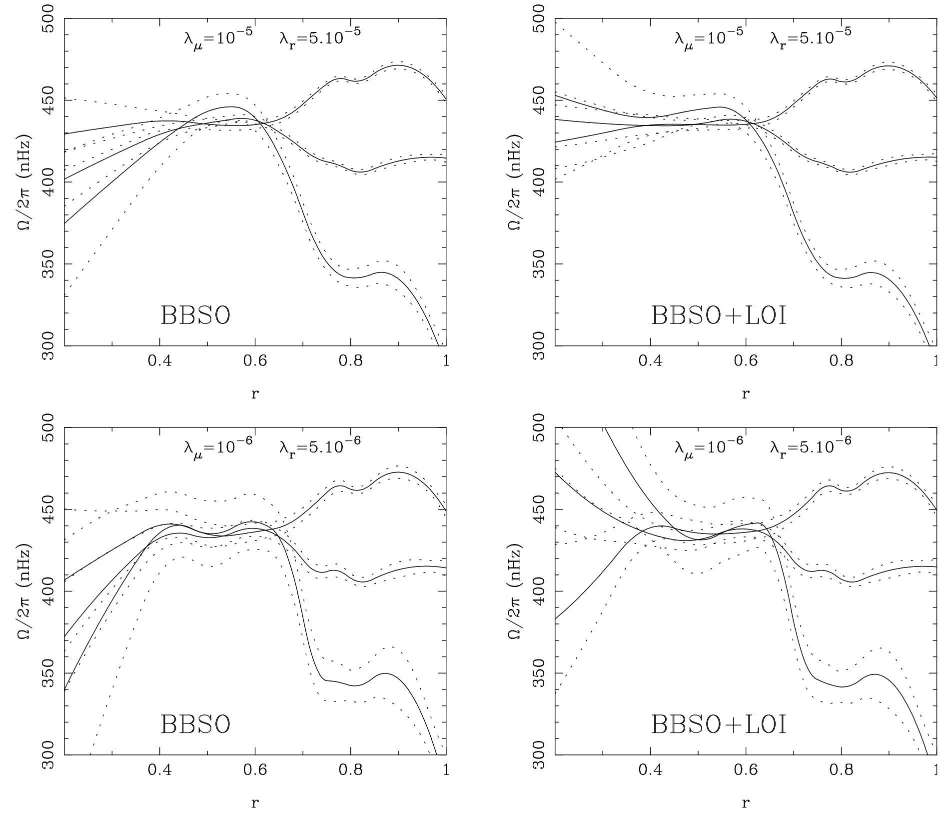

Figure 6: Comparison of the rotation rates obtained with BBSO data and BBSO+LOI data for two values of the regularizing constants λ_r and λ_μ.

and shows that the results are not much sensitive to the order of the splines which are used.

We have applied this code to BBSO and LOI data. Our preliminary computations show that the data BBSO+LOI and their uncertainties do not strongly constrain the solution for $r < 0.4$. However, the test quantities of Table I show that the LOI observations, in agreement with Tomczyk et al (Ref. 10), seems to indicate that the rotation rate remains lower than the surface equatorial rate down to 0.3 $R_\odot$.

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