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Asymptotic analysis of extended magnetohydrodynamics

Nicolas Besse [a], Christophe Cheverry [b,*]

[a] Observatoire de la Côte d'Azur, Bd de l'Observatoire CS 34229, 06304 Nice Cedex 4, France
[b] Institut Mathématique de Rennes, Campus de Beaulieu, 263 avenue du Général Leclerc CS 74205, 35042 Rennes Cedex, France

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Abstract

The derivation of a reduced eXtended MagnetoHydroDynamic (XMHD) model for fusion plasmas is formulated as a singular limit of a hyperbolic-parabolic system implying a large parameter. It is proven that the solutions to the barotropic XMHD system converge to those of a well-posed corresponding reduced model, thereby providing a rigorous justification for its use as a modulation equation for both theoretical and numerical studies. Our analysis involves many new aspects in comparison with well-known results [3,17,20,26,29] about incompressible limits, including the XMHD specificities and, due to the presence of anisotropy, the use of a pseudo-differential symmetrizer whose symbol involves singularities along a line.

Keywords: Hyperbolic-parabolic systems of conservation laws; Singular limits of nonlinear PDEs; Pseudo-differential symmetrizers; Magnetohydrodynamics; Hall and electron inertial effects; Fusion plasmas

Contents

* Corresponding author.
E-mail addresses: nicolas.besse@oca.eu (N. Besse), christophe.cheverry@univ-rennes.fr (C. Cheverry).

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1. Introduction

Plasmas, composed of a rich mixture of charged particles such as heavy protons and light electrons, have long fascinated the scientific community with their complex phenomena. Their study can be approached through various descriptions, including kinetic [8] and fluid [11] models, depending on the context. This work focuses on the two-fluid aspects, specifically within the framework of extended magnetohydrodynamics [1,7,9,19,22,23], a state-of-the-art fluid plasma model that incorporates small-scale features driven by Hall and electron inertial effects. Building on research in fusion plasmas, as in [2,12] but from a different angle, our objective is to investigate the additional impact of a fixed given external magnetic field.

We begin by considering eXtended MagnetoHydroDynamics (XMHD) in the presence of a strong magnetic field. The primary objective is to derive reduced models that can be effectively employed for both theoretical and numerical investigations. In doing so, we build upon and expand the existing approaches outlined in [10,13,14]. Furthermore, this study provides an opportunity to revisit and contribute to the ongoing research on singular limits of partial differential equations (PDEs) with large operators, a topic that has been explored in various works on incompressible limits (see [3,16,17,20,26,28,29] and references therein). By examining the singular limits of XMHD, we aim to provide new insights and advances in this area of research.

The initial value problem for XMHD has been recently solved in the article [4], leading to local well-posedness for smooth H^s-solutions (with $s > 5/2$). Let $\varepsilon \in]0,1]$ with $\varepsilon \ll 1$. In the presence of a strong magnetic field, the large coefficient $1/\varepsilon$ appears in the equations. Then, the task (see Section 2 for explanation) is to identify the behavior when ε goes to zero of the solutions to the following penalized system

$$\left\{\begin{aligned} &\partial_t q + (v\cdot\nabla_\varepsilon) q + \frac{1}{\varepsilon}\, \mathrm{a}(\bar{\mathrm{q}}+\varepsilon\, q)\ \nabla_\varepsilon\cdot v = 0\,,\\ &\partial_t v + (v\cdot\nabla_\varepsilon) v + \frac{1}{\varepsilon}\, \mathrm{a}(\bar{\mathrm{q}}+\varepsilon\, q)\ \nabla_\varepsilon q - \frac{1}{\varepsilon}\,(A^*-A)\times e_z\\ &\qquad -(A^*-A)\times(\nabla_\varepsilon\times A^*) + \nabla_\varepsilon\left(\frac{|A^*-A|^2}{2}\right) = \nu\,\nabla_\varepsilon(\nabla_\varepsilon\cdot v)\,,\\ &\partial_t A^* - \big(v-(A^*-A)\big)\times(\nabla_\varepsilon\times A^*) - (A^*-A)\times(\nabla_\varepsilon\times v)\\ &\qquad -\frac{1}{\varepsilon}\,\big(v-(A^*-A)\big)\times e_z + \frac{1}{\varepsilon}\,\nabla_\varepsilon l = 0\,, \end{aligned}\right. \tag{1.1}$$

completed with the divergence-free condition

$$\frac{1}{\varepsilon}\,\nabla_\varepsilon\cdot A^* = 0\,, \tag{1.2}$$

and with a constitutive relation specifying the link between A and A^* through

$$\frac{1}{\varepsilon}\left(\mathrm{g}^{-1}(\bar{\mathrm{q}}+\varepsilon\, q)\, A^{*}-\mathrm{g}^{-1}(\bar{\mathrm{q}}+\varepsilon\, q)\, A-\nabla_{\varepsilon}\times(\nabla_{\varepsilon}\times A)\right)=0\,. \tag{1.3}$$

At the level of (1.2) and (1.3), the weight ε^{-1} may seem superfluous. It is put here in factor to obtain a convenient hierarchy when performing the WKB analysis. We fix some bulk viscosity $\nu\in\mathbb{R}_{+}$. The content of (1.1)-(1.2)-(1.3) is analyzed in [4]. For $\nu\in\mathbb{R}_{+}^{*}$, it can be viewed as a hyperbolic-parabolic system [18]. As explained in [4], in the compressible context (1.1), the presence of ν is needed to recover the local well-posedness.

Here, $\mathrm{g}:\mathbb{R}_{+}^{*}\to\mathbb{R}_{+}^{*}$ is a smooth given strictly increasing function, with inverse g^{-1}; the value $\bar{\mathrm{q}}$ is issued from a prescribed constant density $\bar{\rho}\in\mathbb{R}_{+}^{*}$ through $\bar{\mathrm{q}}=\mathrm{g}(\bar{\rho})$; the function $\mathrm{a}:\mathbb{R}_{+}^{*}\to\mathbb{R}_{+}^{*}$ is deduced from g through (2.6); $t\in\mathbb{R}$ represents the time variable; $(x,y)\in\mathbb{R}^{2}$ or $(x,y)\in\mathbb{T}^{2}$ are the horizontal space coordinates; $z\in\mathbb{R}$ or $z\in\mathbb{T}$ is the vertical coordinate, while $e_{z}={}^{t}(0,0,1)$ is the vertical direction, along which the strong magnetic field is set.

We denote by Ω the spatial domain, which is therefore either $\mathbb{R}^{3}$, $\mathbb{R}^{2}\times\mathbb{T}$, $\mathbb{T}^{2}\times\mathbb{R}$ or $\mathbb{T}^{3}$. The differential operators

$$\nabla_{\varepsilon}:={}^{t}(\partial_{x},\partial_{y},\varepsilon\,\partial_{z})\,,\qquad \Delta_{\varepsilon}:=\nabla_{\varepsilon}\cdot\nabla_{\varepsilon}=\partial_{x}^{2}+\partial_{y}^{2}+\varepsilon^{2}\,\partial_{z}^{2}\,,$$

stand for anisotropic versions of the gradient and the Laplacian. The unknown is $U\equiv U_{\varepsilon}$ (the index ε will be dropped when it is not confusing). It is composed of a scalar component $q\equiv q_{\varepsilon}\in\mathbb{R}_{+}^{*}$ (likened to a pressure), a three dimensional velocity $v={}^{t}(v_{x},v_{y},v_{z})\equiv v_{\varepsilon}\in\mathbb{R}^{3}$, a (dynamical) magnetic potential $A^{*}\equiv A_{\varepsilon}^{*}\in\mathbb{R}^{3}$, a corresponding (physical) magnetic potential $A\equiv A_{\varepsilon}\in\mathbb{R}^{3}$, and a corrector $l\equiv l_{\varepsilon}\in\mathbb{R}$. Retain that

$$U_{\varepsilon}(t,x)=(q_{\varepsilon},v_{\varepsilon},A_{\varepsilon}^{*},A_{\varepsilon},l_{\varepsilon})(t,x)\in\mathbb{R}_{+}^{*}\times\mathbb{R}^{3}\times\mathbb{R}^{3}\times\mathbb{R}^{3}\times\mathbb{R}\,. \tag{1.4}$$

The scalar function l plays the part of a Lagrange multiplier, while A can be deduced from the field A^{*} through the relation (1.3). In practice, the unknown is composed of the reduced set of state variables $U^{r}\equiv U_{\varepsilon}^{r}:=(q_{\varepsilon},v_{\varepsilon},A_{\varepsilon}^{*})$. In what follows, the system of equations built with (1.1)-(1.2)-(1.3), imposed on U_{ε} or equivalently U_{ε}^{r}, is written in the abbreviated form $\mathcal{L}(U_{\varepsilon};\partial)U_{\varepsilon}=0$ or $\mathcal{L}(U_{\varepsilon}^{r};\partial)U_{\varepsilon}^{r}=0$.

At time $t=0$, we consider a family $(U_{\varepsilon}^{in})_{\varepsilon}$ of initial conditions indexed by $\varepsilon\in[0,1]$. In other words, with $U^{in}\equiv U_{\varepsilon}^{in}$, we start with

$$U_{\varepsilon}(0,x)=U_{\varepsilon}^{in}(x)=(q_{\varepsilon}^{in},v_{\varepsilon}^{in},A_{\varepsilon}^{in*},A_{\varepsilon}^{in},l_{\varepsilon}^{in})(x)\,. \tag{1.5}$$

We assume that U^{in} is smooth enough with respect to both variables ε and x, namely that

$$U^{in}\in C^{\infty}\big([0,1];H^{s}(\Omega;\mathbb{R}^{11})\big)\,,\qquad s>5/2\,. \tag{1.6}$$

The expression U^{in} must be adjusted in coherence with both (1.2) and (1.3), and also with the condition on $\Delta_{\varepsilon}l^{in}$ which can be extracted by applying the (anisotropic divergence) operator $\nabla_{\varepsilon}\cdot$ on the last equation of (1.1). In particular, in view of (1.2), we must impose

$$\nabla_{\varepsilon}\cdot A_{\varepsilon}^{in*}\equiv 0\,,\qquad \forall\varepsilon\in[0,1]\,. \tag{1.7}$$

Moreover, for simplicity, we consider only *prepared data* in the sense of Definition 3. This implies that $\partial_t U_\varepsilon(0,\cdot)$ stays bounded as $\varepsilon \to 0$. In other words

$$\sup \left\{ \| \, \partial_t U_\varepsilon(0,\cdot) \, \|_{H^s} \, ; \, \varepsilon \in]0,1] \right\} < +\infty \,. \tag{1.8}$$

Then, following the terminology of [28], the nature of the singular limit may be called *slow* since rapid time variations are avoided (at least at time $t = 0$).

The origin of (1.1)-(1.2)-(1.3) is thoroughly discussed in Section 2. The motivation is twofold. These equations appear when studying in the context of XMHD the propagation of high frequency waves (in accordance with the viewpoint of nonlinear geometric optics), see Subsection 2.1. They also come from large aspect ratio considerations relevant to fusion devices (such as tokamaks), see Subsection 2.2. Notably, they have direct implications for practical applications in the field of conducting fluids, highlighting the significance of the actual theoretical framework.

The mathematical setup is rooted in the pioneering works of Klainermann-Majda [20] and Kreiss [21]. Subsequently, Métivier [25], Rauch [27], Schochet [29] and numerous other researchers have approached such asymptotic issues from diverse perspectives.

A comprehensive overview of the various contributions can be found in the review article [3] by Alazard. The study of (1.1) is in that long tradition, while investigating completely new and intriguing aspects:

- Firstly, the equations listed inside (1.1) take into account the Hall (or ion inertial) effects (as in [15] and references therein) but they also incorporate electron inertial phenomena, leading to some unconventional form of the quasilinear part which can be detected by comparing (1.1) with MHD equations.
- Secondly, the content of the penalization terms, those which inside (1.1) involve the singular weight $1/\varepsilon$, deviates from the usual structures and does not match directly with a skew-adjoint property.
- Thirdly, the presence of anisotropy in the spatial variable x complicates matters.

The combination of these three distinct features leads to a multitude of novel mathematical challenges that we will solve.

Fix any $\nu \in \mathbb{R}_+^*$. As shown in [4], for all $\varepsilon \in]0,1]$, there exists a maximal positive time of existence, denoted by $T_\varepsilon \equiv T_\varepsilon(\nu) \in \mathbb{R}_+^*$, which gives rise on the interval $[0, T_\varepsilon[$ to a unique solution $U_\varepsilon^r = {}^t(q_\varepsilon, v_\varepsilon, A_\varepsilon^*) \in C\big([0, T_\varepsilon[; H^s(\Omega)\big)$. This preliminary information can be supplemented in the following way.

Theorem 1 *(Uniform lifespan and main asymptotic behavior). Fix any $\nu \in \mathbb{R}_+^*$. Let U^{in} be as in* (1.6)*. Assume that the family $(U_\varepsilon^{in})_\varepsilon$ is prepared in the sense of Definition 3. Then, the lifespan T_ε is uniformly bounded below by a positive constant (depending on ν but not on $\varepsilon \in]0,1]$). More precisely:*

$$\exists \, T \in \mathbb{R}_+^* \, ; \qquad T \equiv T(\nu) \sim \nu \,, \quad 0 < T \leq T_\varepsilon \,, \quad \forall \varepsilon \in]0,1] \,. \tag{1.9}$$

Let $U_0^r = {}^t(q_0, v_0, A_0^*)$ be the profile which, restricting T if necessary, is uniquely determined on $[0, T]$ by solving (as indicated in Proposition 4) the modulation equation which is built with (3.8)-(3.9)-(3.10)-(3.11) *and which is associated with the initial data U_0^{in}. Then, for all $s > 5/2$, we can find a constant $C(s) \in \mathbb{R}_+$ such that*

$$\| q_\varepsilon - q_0 \|_{L^\infty([0,T];H^s(\Omega))} + \| v_\varepsilon - v_0 \|_{L^\infty([0,T];H^s(\Omega))} \le C(s)\,\varepsilon\,. \tag{1.10}$$

Moreover, introducing the two magnetic fields $B_\varepsilon := \nabla_\varepsilon \times A_\varepsilon$ and $B_0 := {}^t({}^t\nabla_\perp, 0) \times A_0$, we find that

$$\| B_\varepsilon - B_0 \|_{L^\infty([0,T];H^{s-1}(\Omega))} \le C\,\varepsilon\,. \tag{1.11}$$

The presence of the large factor ε^{-1} in the system (1.1) renders numerical simulations computationally demanding. To address this challenge, the concept of singular limits [3,17,20,26,28,29] offers a viable solution by approximating complex systems with more robust models that are independent of ε. This approach has been extensively explored in the context of MagnetoHydroDynamics (MHD), as seen in [13,14,16,17,20]. More recently, similar efforts have been initiated in the realm of XMHD, focusing on specific scenarios such as X-point collapse [2] and specialized configurations [12]. In view of Theorem 1, this goal can be achieved in the general framework of (1.1).

Although the local smooth existence theory for (1.1) was established in [4], this result alone is insufficient in the context of (1.1)-(1.2)-(1.3). Indeed, a crucial aspect to address is the potentially destabilizing impact of large factors, such as $1/\varepsilon$, on the energy estimates. For further insight into the origin of this issue, we refer the reader to Section 2.4. Note that the asymptotic analysis underlying Theorem 1 diverges from conventional approaches in several respects.

Theorem 1 encompasses two key aspects: the uniform lifespan $T \in \mathbb{R}_+^*$ and the stability issue which is addressed in (1.10)-(1.11). Both of these results directly follow from the construction of a convenient symmetrizer.

By exhibiting a suitably chosen pseudo-differential symmetrizer, we can establish that the penalized terms in (1.1) do not induce destabilizing amplifications. Nevertheless, the symmetrizer's symbol exhibits a complex dependence on ε, precluding a simple power series expansion and introducing additional subtleties.

Access to B_ε provides the effective magnetic field, rendering (1.11) physically meaningful. In contrast, Theorem 1 does not ensure that the difference $A_\varepsilon^* - A_0^*$ becomes small when ε goes to zero. As explained in Paragraph 4.1.1, an amplification factor of $1/\varepsilon$ can persist at the level of A^*. Furthermore, we claim that a more precise expansion of U_ε in powers of ε cannot be made feasible. These are limitations which suggest the presence of fundamental underlying difficulties.

Indeed, the pertinent (pseudo-differential) symmetrizer is not uniformly invertible across the entire phase space. More precisely, its inverse exhibits a singularity with respect to ε along a set of measure zero (the line $\xi_\perp = 0$). This is a consequence of the anisotropic features of the penalized operator ∇_ε. This introduces (in addition to those coming from the XMHD context) numerous complications, which we will initially address using the simplified model (4.1)-(4.2) derivated from the original system (1.1)-(1.2)-(1.3).

The paper is organized as follows. Section 2 provides a motivation for studying the system of equations (1.1)-(1.2)-(1.3) by highlighting the underlying physical and mathematical reasons. Section 3 presents the construction of certain approximate solutions, valid up to the order 1 (see Definition 2), denoted by $U_0 + \varepsilon\,U_1$; it describes the modulation equation on U_0^r and the procedure for solving it. Section 4 serves as an introduction to the subject by analyzing a toy model, providing preliminary tools and insights; it reveals the crucial role of symmetrizers; it also highlights the fundamental limitations of attempting a complete expansion up to any order N. The paper culminates in Section 5 with a focus on the exact solution U_ε and the proof of the error estimates (1.10) and (1.11).

2. Motivations, modeling and difficulties

The equations of XMHD that are highlighted by physicists [1,7,9,19,22,23] are based on a Lagrangian specification of the flow field. The corresponding Eulerian formulation involves state variables which are the density $\rho \in \mathbb{R}_+$, the velocity $\mathrm{v} \in \mathbb{R}^3$, the magnetic field $\mathrm{B} \in \mathbb{R}^3$ and its dynamical counterpart $\mathrm{B}^* \in \mathbb{R}^3$.

The XMHD equations take the form

$$\begin{cases} \partial_t \rho + (\mathrm{v}\cdot\nabla)\rho + \rho\,\nabla\cdot\mathrm{v} = 0\,, \\ \partial_t \mathrm{v} + (\mathrm{v}\cdot\nabla)\mathrm{v} + \dfrac{\nabla \mathrm{p}(\rho)}{\rho} + \mathrm{B}^* \times \dfrac{\nabla\times\mathrm{B}}{\rho} + d_e^2\,\nabla\Big(\dfrac{|\nabla\times\mathrm{B}|^2}{2\,\rho^2}\Big) = \nu\,\nabla(\nabla\cdot\mathrm{v})\,, \\ \partial_t \mathrm{B}^* + \nabla\times\Big(\mathrm{B}^*\times\Big(\mathrm{v} - d_i\,\dfrac{\nabla\times\mathrm{B}}{\rho}\Big)\Big) + d_e^2\,\nabla\times\Big((\nabla\times\mathrm{v})\times\dfrac{\nabla\times\mathrm{B}}{\rho}\Big) = 0\,, \end{cases} \tag{2.1}$$

together with the constitutive relation

$$\mathrm{B}^* = \mathrm{B} + d_e^2\,\nabla\times\Big(\frac{\nabla\times\mathrm{B}}{\rho}\Big). \tag{2.2}$$

The map $\mathrm{p} : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is a given strictly increasing state function. Let $\bar\rho \in \mathbb{R}_+^*$ be a fixed constant pressure. Since $\mathrm{p}' > 0$, instead of working with ρ, we can alternatively deal with the variable q given by

$$\mathrm{q} := \mathrm{g}(\rho)\,, \qquad \mathrm{g}(\rho) := \int_{\bar\rho}^{\rho} \frac{\sqrt{\mathrm{p}'(s)}}{s}\,ds\,, \qquad \mathrm{g} : \mathbb{R}_+^* \to \mathbb{R}_+^*\,.$$

The advantage of using q instead of ρ is to gain some quasilinear symmetry at the level of the two first equations of (2.1). The coefficient $\nu \in \mathbb{R}_+$ is to introduce some bulk (fluid) viscosity. In accordance with Gauss's law for magnetism, the field B must satisfy $\nabla\cdot\mathrm{B} = 0$, so that $\nabla\cdot\mathrm{B}^* = 0$. Let A^* be the field determined by the following div-curl problem

$$d_e\,\nabla\times\mathrm{A}^* = \mathrm{B}^*\,, \qquad \nabla\cdot\mathrm{A}^* = 0\,, \tag{2.3}$$

with zero condition at infinity in the global case and mean zero in the periodic case. From the field A^*, following [4] and knowing that ρ is bounded and stay away from zero, we can deduce A by inverting the relation

$$\mathrm{A}^* = \mathrm{A} + d_e^2\,\frac{\nabla\times(\nabla\times\mathrm{A})}{\rho}\,. \tag{2.4}$$

This allows to define a magnetic potential A associated with B through $\nabla\times\mathrm{A} = \mathrm{B}$. Observe that, due to the variations of ρ, we do not have necessarily $\nabla\cdot\mathrm{A} = 0$.

The two dimensionless parameters d_e and d_i represent respectively the normalized electron and ion skin depths; they are independent though in practice they are often adjusted in such a way that $0 \le d_e \le d_i \ll 1$. Briefly:

– For $d_e = d_i = 0$, we find the equations of compressible magneto-hydrodynamics, called *ideal MHD*. In this somewhat classical context [24], singular limits have been thoroughly investigated [25–29], see [17] for recent developments;
– For $0 = d_e < d_i$, we are faced with *Hall MHD* which has been extensively studied over the past ten years and which is known today as being unstable (even in the presence of a fluid viscosity), see [15] and references therein;
– For $0 < d_e \leq d_i$ and $0 < \nu$, we meet *eXtended MHD* which, due to remarkable stabilizing properties occurring at high frequencies, becomes again well-posed [4].

For these reasons, in what follows, we assume that $0 < d_e \leq d_i$ and $0 < \nu$. By this way, we incorporate both the Hall effect (related to the term with d_i in factor) and the electron inertial effects (which are driven by the terms with d_e^2 in factor). As explained in [4], the formulation (2.1) is not suitable. To recover (locally in time) energy estimates, it must be transformed into the *potential formulation*. This requires to use A^* instead of B^* and q instead of ρ. This also entails a convenient rescaling procedure (to normalize d_e and make appear the role of the ratio $d := d_i/d_e$) which is detailed in [4].

The rescaled version of (2.1), which is amenable to energy estimates [4], takes the form

$$\begin{cases} \partial_t \mathrm{q} + (\mathrm{v}\cdot\nabla)\mathrm{q} + \mathrm{a}(\mathrm{q})\ \nabla\cdot\mathrm{v} = 0\,, \\ \partial_t \mathrm{v} + (\mathrm{v}\cdot\nabla)\mathrm{v} + \mathrm{a}(\mathrm{q})\ \nabla \mathrm{q} \\ \qquad -(\mathrm{A}^*-\mathrm{A})\times(\nabla\times\mathrm{A}^*) + \nabla\Big(\dfrac{|\mathrm{A}^*-\mathrm{A}|^2}{2}\Big) = \nu\ \nabla(\nabla\cdot\mathrm{v})\,, \\ \partial_t \mathrm{A}^* - \big(\mathrm{v} - d\,(\mathrm{A}^*-\mathrm{A})\big)\times(\nabla\times\mathrm{A}^*) - (\mathrm{A}^*-\mathrm{A})\times(\nabla\times\mathrm{v}) + \nabla \mathrm{l} = 0\,, \end{cases} \tag{2.5}$$

where

$$\mathrm{a}(\mathrm{q}) := \mathrm{g}^{-1}(\mathrm{q})\ \mathrm{g}'\circ\mathrm{g}^{-1}(\mathrm{q})\,, \qquad \mathrm{a}:\mathbb{R}_+^*\to\mathbb{R}_+^*\,, \tag{2.6}$$

and where the scalar function $\mathrm{l}:\mathbb{R}\times\mathbb{R}^3\to\mathbb{R}$ stands for a Lagrangian multiplier which serves to ensure that A^* remains divergence free. On the other hand, the field A can be deduced from A^* through

$$\mathrm{A}^* = \mathrm{A} + \frac{\nabla\times(\nabla\times\mathrm{A})}{\rho}\,. \tag{2.7}$$

This means that the effective unknown is made of the reduced set of state variables $\mathrm{U}^r = (\mathrm{q},\mathrm{v},\mathrm{A}^*)$. To simplify, from now on, we work with the coefficient $d=1$. Without limiting the generality of the following, we can assume that $\bar\rho = 1$, so that $\bar{\mathrm{q}} = \mathrm{g}(1)$. We introduce the (positive) coefficient $\bar{\mathrm{a}} := \mathrm{a}(\bar{\mathrm{q}})$.

The system (2.5)-(2.7) is locally well-posed in H^s for all $s > 5/2$. In connection with applications, of special interest is what happens near a constant magnetic field $\bar{\mathrm{B}}\in\mathbb{R}^3$. With no loss of generality, we can always assume that $\bar{\mathrm{B}} = e_z = {}^t(0,0,1)$. With this in mind, let $\bar{\mathrm{A}}^*$ be a magnetic potential which is adjusted in such a way that $\nabla\times\bar{\mathrm{A}}^* = e_z$ where e_z is the vertical direction. For instance, take for $\bar{\mathrm{A}}^*$ the (non constant) vector field $\bar{\mathrm{A}}^*(\mathrm{x}) = {}^t(0,\mathrm{x},0)$. Define $\bar{\mathrm{q}} := \mathrm{g}(\bar\rho)$, and remark that

$$\bar{\mathrm{U}}^r(\mathrm{x}) := \big(\bar{\mathrm{q}}, 0, \bar{\mathrm{A}}^*(\mathrm{x})\big), \qquad \bar{\mathrm{A}}(\mathrm{x}) \equiv \bar{\mathrm{A}}^*(\mathrm{x}), \qquad \mathrm{l} \equiv 0, \tag{2.8}$$

is a special solution to (2.5)-(2.7). Observe by the way that $\bar{\mathrm{U}}^r$ expressed in terms of $(\rho, \mathrm{v}, \mathrm{B})$ gives rise to the special solution $(\bar{\rho}, 0, \bar{\mathrm{B}})$ of (2.1)-(2.2) with $\mathrm{B} = \bar{\mathrm{B}} = e_z$. This is also a particular solution of Hall MHD ($d_e = 0$) and ideal MHD ($d_e = d_i = 0$). Now, assuming that d_e and d_i are positive, we look for more general solutions which appear as small perturbations around such (universal) stationary solutions $\bar{\mathrm{U}}^r(\mathrm{x})$.

To this end, we seek solutions to (2.5)-(2.7) in the form[1]

$$\mathrm{U} = (\mathrm{q}, \mathrm{v}, \mathrm{A}^*) = \bar{\mathrm{U}}^r + \varepsilon\, U^r, \qquad U^r = (q, v, A^*).$$

Along the same lines, we set $\mathrm{A} = \bar{\mathrm{A}} + \varepsilon\, A$ and $\mathrm{l} = \varepsilon\, l$. By this way, we recover all the components of $U \equiv U_\varepsilon$ with U as in (1.4) and $0 < \varepsilon \ll 1$.

2.1. Nonlinear optics

Conducting fluids are traversed by electromagnetic waves, which can interact with the medium in various ways. These phenomena can be modeled by adjusting the dimensionless parameters to account for special regimes, and by incorporating (high frequency) oscillating source terms or equivalently (high frequency) oscillating initial data into the equations. With this in mind, we take $d_e = d_i = \varepsilon$ (which results in $d = 1$) and we impose $\mathrm{v} = \varepsilon^2\, \nu$. This last choice allows the bulk viscosity to accommodate the propagation of oscillating waves with wavelengths approximately ε.

At such high frequencies, the differential part of (2.4) is more dominant, and ion and electron inertial effects become significant. With this understanding, we look for solutions like

$$\bar{\mathrm{U}}^r\Big(\frac{\mathrm{x}}{\varepsilon}\Big) + \varepsilon\, U^r\Big(t, \frac{\mathrm{x}}{\varepsilon}, \frac{\mathrm{y}}{\varepsilon}, \mathrm{z}\Big), \quad \mathrm{A} = \bar{\mathrm{A}}\Big(\frac{\mathrm{x}}{\varepsilon}\Big) + \varepsilon\, A\Big(t, \frac{\mathrm{x}}{\varepsilon}, \frac{\mathrm{y}}{\varepsilon}, \mathrm{z}\Big), \quad \mathrm{l} = \varepsilon\, l\Big(t, \frac{\mathrm{x}}{\varepsilon}, \frac{\mathrm{y}}{\varepsilon}, \mathrm{z}\Big), \tag{2.9}$$

where $(U^r, A, l)(t, \cdot)$ depends on $(x, y, z) \in \mathbb{T}^2 \times \mathbb{R}$. For the sake of simplicity, we do not take into account the presence of slow variables $(\varepsilon\, x, \varepsilon\, y, \varepsilon\, z)$. On the other hand, we allow anisotropic oscillations in the horizontal plane. At the level of (2.3), (2.4) and (2.1), this means to substitute $\varepsilon^{-1}\, \nabla_\varepsilon$ for ∇. By this way, we find (1.1), (1.2) and (1.3). Observe that the background solution involves a magnetic field which is adjusted such that

$$\nabla \times \Big[\bar{\mathrm{A}}\Big(\frac{\mathrm{x}}{\varepsilon}\Big)\Big] = \frac{1}{\varepsilon}\, (\nabla \times \bar{\mathrm{A}})\Big(\frac{\mathrm{x}}{\varepsilon}\Big) = \frac{1}{\varepsilon} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Thus, we look at what happens near a fixed large constant magnetic field. The perturbation is of small amplitude $\varepsilon \ll 1$ and of high frequency $(1 \ll 1/\varepsilon)$. It belongs to the so called *weakly nonlinear* geometric optics regime. By this way, the discussion falls under the scope of a long tradition of works about strongly magnetized plasmas (see [8] and references therein), with new features issued from the anisotropic and inertial effects.

[1] We will frequently make transformations of the form $\mathrm{W} = \bar{\mathrm{W}} + \varepsilon\, W$ with the new variable W represented by the same letter as W but in a italic font instead of a roman font.

2.2. *Large aspect ratio theory*

The *straight* tokamak model involves a high vertical periodic cylinder of length $2\pi R_0$, whose section is a small horizontal disk of radius a. The corresponding geometry, coordinate system and scalings are detailed carefully in the text of Guillard [13], see Subsection 3.2. The inverse of the aspect ratio, given by $\varepsilon := a/R_0$, can be viewed as a small parameter. Here, we impose $\mathrm{v} = \varepsilon\, \nu$. Starting from the system of equations (2.3)-(2.4)-(2.5), passing in normalized space variables and changing the current time τ into $\tau = a\, t/(v_A \varepsilon)$, with the Alfvén velocity $v_A = \bar{\mathrm{B}}^2/\bar{\rho}$, we recover (1.1)-(1.2)-(1.3). We will not write out this procedure which is a straightforward repetition of [13].

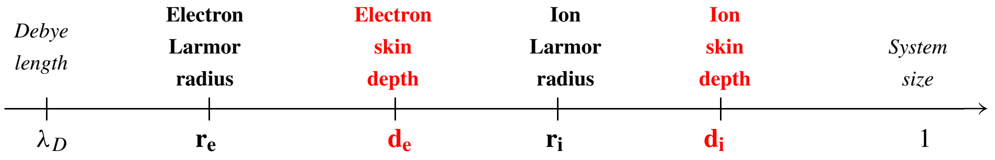
2.3. *Adjustment of parameters*

Subsections 2.1 and 2.2 are aimed at insisting on the *anisotropic* features induced by the geometry of devices and/or by the external (strong) magnetic field $\bar{\mathrm{B}}$. At the same time, the presence of $\bar{\mathrm{B}}$ induces *fast rotations*. From this standpoint, the discussion falls within the scope of rotating frameworks. In our context, both anisotropy and rapid variations in the direction orthogonal to the magnetic field lines are present, with correlated scalings. This topic has been extensively studied in monofluid contexts like the Vlasov-Maxwell equations (gyrokinetic theory [8]) and the MHD equations (see [6,16] and references therein).

The interest of (1.1) is to extend the discussion in a situation where important two-fluid corrections (issued from Hall and electron inertial effects) are incorporated (as a background). This is done here by resorting to XMHD equations instead as usual to the rough model of MHD equations. Plamas produced in fusion devices involve many parameters among which:

- The skin depths d_e and d_i of respectively electrons and ions. As explained in [4,5], the lengths d_e and d_i take into account two-fluid effects. They have been normalized to one at the level of (2.5).
- The Larmor radii r_e and r_i of respectively electrons and ions. They come from the implementation of a strong external magnetic field which generates rapid variations (due to the rotations of particles) occurring at the size of r_e (for electrons) and r_i (for ions).

Typically, in a tokamak, these parameters are adjusted as indicated below:



Our configuration corresponds to the case $r_e \ll d_e \simeq r_i \simeq d_i$ so that $\varepsilon = r_e/d_e \ll 1$. Expressed in terms of the beta parameters β_s, which are equivalent to $(r_s/d_s)^2$, this means that $\beta_e \simeq \varepsilon^2 \ll \beta_i \simeq 1$. The introduction of the small ratio ε is aimed at taking into account both anisotropic effects and rapid rotations of electrons around the field lines.

2.4. *About the construction of a symmetrizer*

We conclude this section by insisting on a difficulty which is inherent in the study of (1.1)-(1.2)-(1.3). The problem comes from a penalization along a non constant solution. To understand why, let us consider a quasilinear symmetric system (the hyperbolic-parabolic situation is similar) of N equations in dimension d (with $\mathrm{x} \in \mathbb{R}^{\mathrm{d}}$), which has the following form

$$S_0(\mathrm{U})\,\partial_t \mathrm{U} + \sum_{j=1}^{\mathrm{d}} S_j(\mathrm{U})\,\partial_j \mathrm{U} = 0\,, \qquad S_j(\mathrm{U}) = {}^t S_j(\mathrm{U})\,. \tag{2.10}$$

We assume that $\bar{\mathrm{U}}(\mathrm{x})$ is a special stationary solution of (2.10) satisfying

$$S_j\big(\bar{\mathrm{U}}(\mathrm{x})\big) = \bar{S}_j(\mathrm{x}) = {}^t\bar{S}_j(\mathrm{x}) \in \mathcal{M}_N(\mathbb{R})\,, \qquad \forall j \in \{1, \cdots, \mathrm{d}\}\,.$$

Now, in line with (2.9), we can seek solutions U of (2.10) in the form

$$\mathrm{U}_\varepsilon(t,\mathrm{x}) = \bar{\mathrm{U}}\Big(\frac{\mathrm{x}}{\varepsilon}\Big) + \varepsilon\, U\Big(t, \frac{\mathrm{x}}{\varepsilon}\Big)\,, \qquad 0 < \varepsilon \ll 1\,.$$

Changing x into $x = \mathrm{x}/\varepsilon$, it is easy to check that U is subject to

$$\begin{aligned} S_0(\bar{\mathrm{U}} + \varepsilon\, U)\,\partial_t U + \frac{1}{\varepsilon}\sum_{j=1}^{\mathrm{d}} \bar{S}_j\,\partial_j U + \sum_{j=1}^{\mathrm{d}} \check{S}_j(\varepsilon, t, x, U)\,\partial_j U \\ + \frac{1}{\varepsilon}\sum_{j=1}^{\mathrm{d}} (U\cdot\nabla_{\mathrm{U}}) S_j(\bar{\mathrm{U}})\,\partial_j \bar{\mathrm{U}} = 0\,, \end{aligned}$$

where the matrices $\check{S}_j$ are smooth with respect to (ε, t, x, U), and with ε in the closed interval [0, 1]. Since the $\bar{S}_j$ and $\check{S}_j$ are symmetric, the first line gives rise to uniform L^2-energy estimates. When $\bar{\mathrm{U}}$ is constant, the same applies to the second line, which simply disappears. Then, the framework is as in [25,27]. But when $\nabla_x \bar{\mathrm{U}} \not\equiv 0$, the last term is likely to be significant. This is exactly what occurs in the context of (2.5).

As shown in [4], resorting to the potential formulation (2.5) is essentially unavoidable. But, in the presence of a non zero magnetic field, a non constant potential field $\bar{\mathrm{A}}^*(x)$ appears. Typically, we have $\nabla_x \times \bar{\mathrm{A}}^* \not\equiv 0$ so that $\nabla_x \bar{\mathrm{U}} \not\equiv 0$. Then, the linearization procedure along $\bar{\mathrm{A}}^*$ generates a non zero second line. It is not excluded, far from it, that the symmetrizers of the first line are inconsistent with a skew-symmetry property of the second line. In fact, as a rule, this is the case.

One of our key contributions is to demonstrate that, in the XMHD framework, such incompatibility can be bypassed. In practice, it is not possible to derive energy estimates directly at the level of (1.1)-(1.2). Instead, we have to find an adequate symmetrizer. The complete construction is achieved in Section 5. It relies on the introduction of a scalar function ψ_ε, see (4.12), which appears naturally when performing in Section 4 the Fourier analysis on a toy model. We find that $\psi_\varepsilon = \psi_0 + O(\varepsilon)$. The role of ψ_0 is essential in next Section 3 to show the existence of solutions to the modulation equations (3.8). For instance, the reader can observe that the relation (3.23) which involves ψ_0 is indispensable for eliminating (the Lagrange multiplier) $v_{1\perp}$ when performing energy estimates.

3. Approximate solutions of order one

The first step in the analysis is to propose a candidate U^a_ε for describing the asymptotic behavior of U_ε when ε goes to zero. Let $j \in \mathbb{N}$. We denote by $\mathcal{C}^j_b \equiv \mathcal{C}^j_b(\Omega)$ the space of j times continuously differentiable functions on the open set Ω, with bounded derivatives. The Banach space $\mathcal{C}^j_b$ is equipped with its usual norm. Define $\mathcal{C}^\infty_b := \cap\{\mathcal{C}^j_b;\, j \in \mathbb{N}\}$. Recall that the system (1.1)-(1.2)-(1.3) is denoted by $\mathcal{L}(U_\varepsilon; \partial)U_\varepsilon = 0$.

Definition 2 *(Approximate solution). Let $N \in \mathbb{N}$ and $T \in \mathbb{R}^*_+$. Consider the expansion*

$$U^a_\varepsilon = U_0 + \varepsilon\, U_1 + \cdots + \varepsilon^N\, U_N\,, \qquad U_j = (q_j, v_j, A^*_j, A_j, l_j) \in \mathcal{C}^\infty_b([0,T] \times \Omega)\,. \tag{3.1}$$

We say that U^a_ε is an approximate solution of order N on the time interval $[0,T]$ to the system (1.1)-(1.2)-(1.3) when $\mathcal{L}(U^a_\varepsilon; \partial)U^a_\varepsilon = R^a_\varepsilon$ with a reminder R^a_ε satisfying

$$\forall\, j \in \mathbb{N}\,, \qquad \exists\, C_j \in \mathbb{R}_+\,, \qquad \sup_{\varepsilon \in]0,1]} \parallel R^a_\varepsilon \parallel_{\mathcal{C}^j_b([0,T]\times\Omega)} \le C_j\, \varepsilon^N\,. \tag{3.2}$$

Remark that

$$\nabla_\varepsilon = {}^t({}^t\nabla_\perp, 0) + \varepsilon\, {}^t(0,0,\partial_z)\,, \qquad \nabla_\perp := {}^t(\partial_x, \partial_y)\,.$$

By this way, the two first components of the velocity v are set aside. Define

$$v_\perp := {}^t(v_x, v_y)\,, \qquad v^\perp_\perp := {}^t(v_y, -v_x)\,, \qquad \nabla_\perp \times v_\perp := \partial_x v_y - \partial_y v_x = \nabla_\perp \cdot v^\perp_\perp\,.$$

3.1. Prepared data

The terms which inside (1.1) involve the weight ε^{-1} are able to generate rapid time variations. On the contrary, we would like to focus on slow dynamics. To this end, we need to impose supplementary conditions. As a matter of fact, we can eliminate from (1.1) and (1.2) the singular terms by imposing

$$\nabla_\perp \cdot v_{0\perp} = 0\,, \qquad \nabla_\perp \cdot A^*_{0\perp} = 0\,, \tag{3.3}$$

as well as

$$\begin{cases} \nabla_\perp l_0 = v^\perp_{0\perp} - A^{*\perp}_{0\perp} + A^\perp_{0\perp}\,, \\ \bar{\mathrm{a}}\, \nabla_\perp q_0 = A^{*\perp}_{0\perp} - A^\perp_{0\perp}\,. \end{cases} \tag{3.4}$$

Since $\mathrm{g}^{-1}(\bar{\mathrm{q}}) = 1$, the constitutive relation (1.3) gives $A^*_0 - A_0 = {}^t({}^t\nabla_\perp, 0) \times \big({}^t({}^t\nabla_\perp, 0) \times A_0\big)$. From (3.4), it follows in particular that $\nabla_\perp \cdot A_{0\perp} = 0$, and then

$$A_0 = (\mathrm{Id} - \Delta_\perp)^{-1}\, A^*_0\,. \tag{3.5}$$

The procedure is as follows. We select two vector fields $v_{0\perp}$ and $A^*_{0\perp}$ which are divergence free and smooth (say in $\mathcal{C}^\infty_b$). This implies

$$\nabla_\perp \times (A^{*\perp}_{0\perp} - A^{\perp}_{0\perp}) = -\nabla_\perp \cdot (A^*_{0\perp} - A_{0\perp}) = 0\,, \qquad \nabla_\perp \times v^{\perp}_{0\perp} = -\nabla_\perp \cdot v_{0\perp} = 0\,.$$

With the boundary conditions, this furnishes necessary and sufficient conditions for the existence of functions l_0 and q_0 solving (3.4). More precisely, applying the divergence operator $\nabla_\perp\cdot$ to (3.4), using (3.5) and inverting the Laplacian operator $\Delta_\perp$, we obtain

$$l_0 = \Delta_\perp^{-1}\, \nabla_\perp \times v_{0\perp} + (\mathrm{Id} - \Delta_\perp)^{-1}\, \nabla_\perp \times A^*_{0\perp}\,, \tag{3.6}$$

$$q_0 = -\,\bar{\mathrm{a}}^{-1}\, (\mathrm{Id} - \Delta_\perp)^{-1}\, \nabla_\perp \times A^*_{0\perp}\,. \tag{3.7}$$

Definition 3 *(Prepared initial data). We say that* $(q^{in}, v^{in}, A^{in*})(0,\cdot)$ *is prepared when its components are subject to* (3.3) *and* (3.7)*. By extension, we say that* $U^{in}(0,\cdot)$ *or the family* $\{U^{in}(\varepsilon, x)\}_\varepsilon$ *are prepared when* $(q^{in}, v^{in}, A^{in*})(0,\cdot)$ *is prepared, whereas* $A^{in}(0,\cdot)$ *and* $l^{in}(0,\cdot)$ *are determined from* $(q^{in}, v^{in}, A^{in*})(0,\cdot)$ *by* (3.5) *and* (3.6)*.*

By this way, we are sure that $\partial_t U_\varepsilon(0,\cdot)$ computed from spatial derivatives through (1.1) with $U_\varepsilon(0,\cdot) = U^{in}_\varepsilon$ is uniformly bounded with respect to $\varepsilon \in]0,1]$. Thus, prepared data are good candidates for solving (1.1) without implementing rapid time variations.

3.2. *The modulation equations*

Consider a profile U_0 that is adjusted as indicated in Subsection 3.1. Observe that degrees of freedom remain available concerning U_0 since the content of $(\nabla_\perp \times v_{0\perp}, v_{0z}, \nabla_\perp \times A^*_{0\perp}, A^*_{0z})$ has not yet been specified. On the contrary, knowing that $\nabla_\perp \cdot A^*_{0\perp} = 0$, the expressions q_0, A_0 and l_0 may be deduced from $\nabla_\perp \times v_{0\perp}$ and $\nabla_\perp \times A^*_{0\perp}$ through the relations (3.5), (3.6) and (3.7).

Now, let us fix any prepared initial condition U^{in} as in (1.6). Given $T \in \mathbb{R}^*_+$ small enough, the purpose is to determine from $U^{in}(0,\cdot)$ an approximate solution of order 1, denoted by $U_0 + \varepsilon\, U_1$, satisfying $U_0(0,\cdot) = U^{in}(0,\cdot)$. This will be done through evolution equations called *modulation equations*, derived in Paragraph 3.2.1. This requires to adjust the content of the corrector U_1 adequately, as indicated in Paragraph 3.2.2. The well-posedness of these equations is proved in Paragraph 3.2.3 through energy estimates.

3.2.1. *Formal calculus*

The goal here is to construct an approximate solution of order 1. To this end, we select some $U_1 \in \mathcal{C}^\infty_b$, and we compute

$$\mathcal{L}(U_0 + \varepsilon\, U_1; \partial)(U_0 + \varepsilon\, U_1) = R^a_\varepsilon = R^a_0 + O(\varepsilon)\,.$$

First and foremost, we must impose $R^a_0 = 0$. The content of the remainder R^a_0 is built with contributions issued from (1.1), (1.2) and (1.3). From (1.1), we deduce that

$$\begin{cases} \partial_t q_0 + (v_{0\perp}\cdot\nabla_\perp) q_0 + \bar{\mathrm{a}}\,\nabla_\perp\cdot v_{1\perp} + \mathrm{a}'(\bar{\mathrm{q}})\, q_0\,\nabla_\perp\cdot v_{0\perp} + \bar{\mathrm{a}}\,\partial_z v_{0z} = 0\,, \\ \partial_t v_{0\perp} + (v_{0\perp}\cdot\nabla_\perp) v_{0\perp} + \bar{\mathrm{a}}\,\nabla_\perp q_1 + \mathrm{a}'(\bar{\mathrm{q}})\, q_1\,\nabla_\perp q_0 \\ \qquad - (\nabla_\perp\times A^*_{0\perp})\,(A^*_{0\perp}-A_{0\perp})^\perp - (A^*_{0z}-A_{0z})\,\nabla_\perp A^*_{0z} \\ \qquad + \dfrac{1}{2}\,\nabla_\perp\big(|A^*_0 - A_0|^2\big) - (A^*_{1\perp}-A_{1\perp})^\perp = \nu\,\nabla_\perp(\nabla_\perp\cdot v_{0\perp})\,, \\ \partial_t v_{0z} + (v_{0\perp}\cdot\nabla_\perp) v_{0z} + (A^*_{0\perp}-A_{0\perp})\cdot\nabla_\perp A^*_{0z} + \bar{\mathrm{a}}\,\partial_z q_0 = 0\,, \\ \partial_t A^*_{0\perp} - (\nabla_\perp\times A^*_{0\perp})\,(v_{0\perp}+A_{0\perp}-A^*_{0\perp})^\perp - (v_{0z}+A_{0z}-A^*_{0z})\,\nabla_\perp A^*_{0z} \\ \qquad - (\nabla_\perp\times v_{0\perp})\,(A^*_{0\perp}-A_{0\perp})^\perp - (A^*_{0z}-A_{0z})\,\nabla_\perp v_{0z} \\ \qquad - v^\perp_{1\perp} + (A^*_{1\perp}-A_{1\perp})^\perp + \nabla_\perp l_1 = 0\,, \\ \partial_t A^*_{0z} + (v_{0\perp}+A_{0\perp}-A^*_{0\perp})\cdot\nabla_\perp A^*_{0z} + (A^*_{0\perp}-A_{0\perp})\cdot\nabla_\perp v_{0z} + \partial_z l_0 = 0\,. \end{cases} \tag{3.8}$$

From (1.2), we get easily that

$$\nabla_\perp\cdot A^*_{\perp 1} + \partial_z A^*_{0z} = 0\,. \tag{3.9}$$

From (1.3), we obtain that

$$A^*_{1\perp} - A_{1\perp} + \Delta_\perp A_{1\perp} - \nabla_\perp(\nabla_\perp\cdot A_{1\perp}) = -\,\mathrm{g}'(1)^{-1}\,q_0\,(A^*_{0\perp}-A_{0\perp}) + \nabla_\perp\partial_z A_{0z}\,. \tag{3.10}$$

From (1.3) together with (3.3) and (3.5) to see that $\partial_z(\nabla_\perp\cdot A_{0\perp}) = 0$, we get that

$$A^*_{1z} - A_{1z} + \Delta_\perp A_{1z} = -\,\mathrm{g}'(1)^{-1}\,q_0\,(A^*_{0z}-A_{0z})\,. \tag{3.11}$$

When looking at (3.8), the unknown is $U^r_0 = (q_0, v_0, A^*_0)$, while l_0 must be adjusted in terms of U^r_0 as indicated in (3.6) and A_1 can be deduced (as will be seen later) from U^r_0 through (3.9)-(3.10)-(3.11). The components q_1, l_1 and $v_{1\perp}$ (which occurs two times) must be viewed as correctors to be adjusted conveniently. More precisely, q_1 and l_1 serve to guarantee (3.3) and $v_{1\perp}$ is needed to ensure (3.7), see Subsection 3.2.2. We complete these equations by initial data inherited from (1.5). More precisely, we impose

$$U^r_0(0,\cdot) = (q_0, v_0, A^*_0)(0,\cdot) = (q^{in}, v^{in}, A^{in*})(0,\cdot)\,. \tag{3.12}$$

Proposition 4 *(Local smooth solvability of the modulation equations). Fix any $\nu\in\mathbb{R}_+$. Let $s > 5/2$. Select prepared initial data $(q^{in}, v^{in}, A^{in*})(0,\cdot)$ in $H^s(\mathbb{R}^3)$. There exists a time $T\in\mathbb{R}^*_+$ such that the evolution equation (3.8) with initial condition (3.12) and completed with (3.3)-(3.6)-(3.7)-(3.9)- (3.10)-(3.11) has a unique solution on the interval $[0,T]$, satisfying*

$$U^r_0 = (q_0, v_0, A^*_0) \in \mathcal{C}^j\big([0,T]; H^{s-j}(\mathbb{R}^3)\big)\,, \qquad \forall j\in\mathbb{N},\ \textit{with}\ j < s\,.$$

From U^r_0, we get access to A_0 and l_0 through (3.5) and (3.6). The expression U_0 is therefore identified. When solving (3.8)-(3.9)-(3.10)-(3.11), the content of U^r_1 is determined from this U_0 according to a procedure which is detailed in the next paragraph. From there (working with $s=+\infty$), we can recover an approximate solution $U_0 + \varepsilon\,U_1$ of order 1.

3.2.2. Determination of correctors

We can look at U_1 as a Lagrange multiplier since it allows to guarantee (3.3). At the same time, the components of U_1 take part in the propagation properties and energy estimates. It is therefore important to specify how they are adjusted in terms of U_0. First, we consider A_1^* and A_1. Since the system (3.8) does not involve A_{1z}^* and A_{1z}, we can take $A_{1z} = 0$ and just define A_{1z}^* through (3.11). Then, we remark that $A_{1\perp}$ can be deduced from $A_{1\perp}^*$. Indeed, from (3.10), we can extract a div-curl problem involving $A_{1\perp}$. More precisely, exploiting (3.4) and (3.9), we obtain

$$\begin{cases} \nabla_\perp \cdot A_{1\perp} = -\partial_z A_{0z}\,, \\ \nabla_\perp \times A_{1\perp} = (\mathrm{Id} - \Delta_\perp)^{-1}\, \nabla_\perp \times A_{1\perp}^* + \mathrm{g}'(1)^{-1}\, (\mathrm{Id} - \Delta_\perp)^{-1}\, \nabla_\perp \times \big(q_0\, (A_{0\perp}^* - A_{0\perp})\big)\,. \end{cases}$$

This allows to determine $A_{1\perp}$ as a function of U_0, derivatives of U_0 and $\nabla_\perp \times A_{1\perp}^*$. Retain that we have (3.9), as well as

$$\nabla_\perp \cdot (A_{1\perp}^* - A_{1\perp}) = \Delta_\perp \partial_z A_{0z} = \partial_z (A_{0z} - A_{0z}^*)\,. \tag{3.13}$$

On the other hand, the part $\nabla_\perp \times (A_{1\perp}^* - A_{1\perp})$ can be absorbed by modifying q_1 and l_1. Indeed, for l_1 using the Helmholtz–Hodge decomposition $A_{1\perp}^* - A_{1\perp} = \nabla_\perp \varphi + \nabla_\perp^\perp \Phi$ with $-\Delta_\perp \Phi = \nabla_\perp \times (A_{1\perp}^* - A_{1\perp})$, we obtain $(A_{1\perp}^* - A_{1\perp})^\perp + \nabla_\perp l_1 = \nabla_\perp^\perp \varphi - \nabla_\perp (\Phi - l_1)$ and absorption is obtained by introducting a new l_1, says $\tilde{l}_1$, defined by $\tilde{l}_1 = l_1 - \Phi$. For q_1, absorption is obtained by introducing a new q_1, says $\tilde{q}_1$, defined by $\tilde{q}_1 = q_1 - \Phi$, where Φ is the solution of the following elliptic equation: $\bar{\mathrm{a}} \Delta_\perp \Phi + \mathrm{a}'(\bar{\mathrm{q}}) \nabla_\perp \cdot (\Phi \nabla_\perp q_0) = \nabla_\perp \times (A_{1\perp}^* - A_{1\perp})$. Therefore, Φ or $\nabla_\perp \times (A_{1\perp}^* - A_{1\perp})$ plays no role. For the sake of simplicity, we impose $\nabla_\perp \times (A_{1\perp}^* - A_{1\perp}) = 0$. This can be done by adjusting $\nabla_\perp \times A_{1\perp}^*$ adequately. From there and because $\nabla_\perp \cdot v_{0\perp} = 0$, the divergence of the second equation of (3.8), the one on $v_{0\perp}$, gives rise to the following elliptic equation (on q_1)

$$\begin{aligned} \bar{\mathrm{a}}\, \Delta_\perp q_1 + \mathrm{a}'(\bar{\mathrm{q}})\, \nabla_\perp q_0 \cdot \nabla_\perp q_1 + \mathrm{a}'(\bar{\mathrm{q}})\, \Delta_\perp q_0\, q_1 = &- \mathrm{tr}\,(\nabla_\perp v_{0\perp})^2 \\ &+ \nabla_\perp \cdot \big((\nabla_\perp \times A_{0\perp}^*)\, (A_{0\perp}^* - A_{0\perp})^\perp + (A_{0z}^* - A_{0z})\, \nabla_\perp A_{0z}^* \big) \\ &- \frac{1}{2}\, \Delta_\perp \big(|A_0^* - A_0|^2 \big)\,. \end{aligned} \tag{3.14}$$

Exploiting (3.13) to eliminate the influence of $A_{1\perp}^* - A_{1\perp}$, the rotational of the fourth equation inside (3.8), the one on $A_{0\perp}^*$, gives rise to $\nabla_\perp \cdot v_{1\perp}$. Indeed, we can, in particular, use the first equation inside (3.8), the one on q_0, and equation (3.7) to re-express the term $\partial_t \nabla_\perp \times A_{0\perp}^*$ so as to obtain an elliptic equation for $\nabla_\perp \cdot v_{1\perp}$ with a source term depending only on U_0^r and its spatial x-derivatives. Now, because $\nabla_\perp \cdot A_{0\perp}^* = 0$, the divergence of the fourth equation inside (3.8) reveals a stationary equation which involves $-\nabla_\perp \times v_{1\perp} + \Delta_\perp l_1$ together with expressions depending on U_0^r. It suffices to choose arbitrarily $\nabla_\perp \times v_{1\perp} = 0$, and then to fix $\Delta_\perp l_1$ accordingly.

3.2.3. Proof of Proposition 4

In the perspective of L^2-energy estimates, for reasons clearly explained in Section 4, the system (3.8) is not yet in a suitable form. Instead of working with A_0^*, we need to implement

$$\tilde{A}_0^* = {}^t({}^t \tilde{A}_{0\perp}^*, \tilde{A}_{0z}^*) := \psi_0(D_\perp)\, A_0^*\,, \qquad \tilde{A}_0 = {}^t({}^t \tilde{A}_{0\perp}, \tilde{A}_{0z}) := \psi_0(D_\perp)\, A_0\,, \tag{3.15}$$

where $\psi_0(D_\perp)$ is the Fourier multiplier corresponding to the (bounded) symbol

$$\psi_0(\xi_\perp) := \frac{|\xi_\perp|}{\langle \xi_\perp \rangle} = 1 - \frac{1}{\langle \xi_\perp \rangle \, (|\xi_\perp| + \langle \xi_\perp \rangle)} \,, \qquad \langle \xi_\perp \rangle := (1 + |\xi_\perp|^2)^{1/2}. \tag{3.16}$$

Observe that $\psi_0(\xi_\perp)^{-1}$ goes to $+\infty$ when $|\xi_\perp|$ tends to 0. By contrast, given $j \in \{1,2\}$, the product $\xi_j \, \psi_0(\xi_\perp)^{-1}$ remains bounded near $\xi_\perp = 0$. This means that the action of $\partial_j \, \psi_0(D_\perp)^{-1}$ is well-defined as a bounded operator from H^{s+1} to H^s. From now on, the new unknown is $\tilde U_0^r = {}^t(q_0, v_0, \tilde A_0^*)$. From (3.5), we deduce that

$$A_0^* - A_0 = \psi_0(D_\perp) \, \tilde A_0^* . \tag{3.17}$$

We apply $\psi_0(D_\perp)$ to the two last equations of (3.8). By this way, we obtain a self-contained system of evolution equations on $\tilde U_0^r$, which is

$$\begin{cases} \partial_t q_0 + (v_{0\perp} \cdot \nabla_\perp) q_0 + \bar{\mathrm{a}} \, \nabla_\perp \cdot v_{1\perp} + \mathrm{a}'(\bar{\mathrm{q}}) \, q_0 \, \nabla_\perp \cdot v_{0\perp} + \bar{\mathrm{a}} \, \partial_z v_{0z} = 0 \,, \\ \partial_t v_{0\perp} + (v_{0\perp} \cdot \nabla_\perp) v_{0\perp} + \bar{\mathrm{a}} \, \nabla_\perp q_1 + \mathrm{a}'(\bar{\mathrm{q}}) \, q_1 \, \nabla_\perp q_0 \\ \qquad - \psi_0(D_\perp) \tilde A_{0\perp}^{*\perp} \, \psi_0(D_\perp)^{-1} \, \nabla_\perp \times \tilde A_{0\perp}^* - \psi_0(D_\perp) \tilde A_{0z}^* \, \psi_0(D_\perp)^{-1} \, \nabla_\perp \tilde A_{0z}^* \\ \qquad + \dfrac{1}{2} \, \nabla_\perp \big(|\psi_0(D_\perp) \, \tilde A_0^*|^2 \big) - (A_{1\perp}^* - A_{1\perp})^\perp = \nu \, \nabla_\perp (\nabla_\perp \cdot v_{0\perp}) \,, \\ \partial_t v_{0z} + (v_{0\perp} \cdot \nabla_\perp) v_{0z} + \psi_0(D_\perp) \, \tilde A_{0\perp}^* \cdot \psi_0(D_\perp)^{-1} \, \nabla_\perp \tilde A_{0z}^* + \bar{\mathrm{a}} \, \partial_z q_0 = 0 \,, \\ \partial_t \tilde A_{0\perp}^* - \psi_0(D_\perp) \big[\big(v_{0\perp} - \psi_0(D_\perp) \tilde A_{0\perp}^* \big)^\perp \, \psi_0(D_\perp)^{-1} \, \nabla_\perp \times \tilde A_{0\perp}^* \big] \\ \qquad - \psi_0(D_\perp) \big[\big(v_{0z} - \psi_0(D_\perp) \tilde A_{0z}^* \big) \, \psi_0(D_\perp)^{-1} \, \nabla_\perp \tilde A_{0z}^* \big] \\ \qquad - \psi_0(D_\perp) \big[\psi_0(D_\perp) \tilde A_{0\perp}^{*\perp} \, \nabla_\perp \times v_{0\perp} \big] - \psi_0(D_\perp) \big[\psi_0(D_\perp) \tilde A_{0z}^* \, \nabla_\perp v_{0z} \big] \\ \qquad - \psi_0(D_\perp) v_{1\perp}^\perp + \psi_0(D_\perp)^2 \tilde A_{1\perp}^{*\perp} + \psi_0(D_\perp) \nabla_\perp l_1 = 0 \,, \\ \partial_t \tilde A_{0z}^* + \psi_0(D_\perp) \big[\big(v_{0\perp} - \psi_0(D_\perp) \tilde A_{0\perp}^* \big) \cdot \psi_0(D_\perp)^{-1} \, \nabla_\perp \tilde A_{0z}^* \big] \\ \qquad + \psi_0(D_\perp) \big[\psi_0(D_\perp) \tilde A_{0\perp}^* \cdot \nabla_\perp v_{0z} \big] + \psi_0(D_\perp) \, \partial_z l_0 = 0 \,. \end{cases} \tag{3.18}$$

The existence theory for quasilinear symmetric systems is well established. The crucial point is to derive L^2-energy estimates for the non linear equation and its corresponding linearized versions (which appear after differentiation). In doing so, minimal regularity assumptions are needed on the coefficients. This is usually achieved by supposing that $\tilde U_0^r$ is bounded in the large H^s-norm with $s > 5/2$. In particular, this a priori estimate means that $\tilde U_0^r$ is Lipschitz. This information is recovered ultimately through the H^s-bounds which can be recovered by energy estimates.

In comparison with models issued from ideal MHD, due to the addition of Hall and electron inertial effects, the systems (3.8) and (3.18) involve specific features. It is therefore important to clearly explain in this new context how L^2-norms of solutions can be controlled.

The focus below is on L^2-estimates for the nonlinear equation (3.18). To make progress, we use however the (shorter and more viewable) notations of (3.8). The corresponding bounds for the linearized version of (3.18) follow the same guidelines. The other arguments (fixed-point iteration, $\cdots$) are quite standard [24]. They will not be repeated. We multiply (3.18) by $(q_0, v_{0\perp}, v_{0z}, \tilde A_{0\perp}^*, \tilde A_{0z}^*)$, and then we integrate on the spatial domain Ω. This furnishes a number of contributions which should be aggregated conveniently, as indicated below, to see cancellations.

• Contributions issued from the **transport part**. This is

$$\frac{1}{2}\,\frac{d}{dt}\parallel \tilde{U}_0^r \parallel_{L^2}^2 + \frac{1}{2}\int_\Omega (v_{0\perp}\cdot\nabla_\perp)(q_0^2+|v_0|^2)\,dx$$
$$-\int_\Omega \tilde{A}_{0\perp}^*\cdot\psi_0(D_\perp)\big[(\nabla_\perp\times A_{0\perp}^*)\,(v_{0\perp}+A_{0\perp}-A_{0\perp}^*)^\perp\big]\,dx$$
$$+\int_\Omega \tilde{A}_{0z}^*\;\psi_0(D_\perp)\big[(v_{0\perp}+A_{0\perp}-A_{0\perp}^*)\cdot\nabla_\perp A_{0z}^*\big]\,dx\,.$$

Since $\nabla_\perp\cdot v_{0\perp}=0$, the first integral disappears. Now, we claim that the second and third integrals can be replaced (modulo a constant times the square of the L^2-norm of $\tilde{A}_0^*$) by

$$\begin{aligned}&-\int_\Omega \tilde{A}_{0\perp}^*\cdot\big[(\nabla_\perp\times\tilde{A}_{0\perp}^*)\,(v_{0\perp}+A_{0\perp}-A_{0\perp}^*)^\perp\big]\,dx\\&+\int_\Omega \tilde{A}_{0z}^*\,\big[(v_{0\perp}+A_{0\perp}-A_{0\perp}^*)\cdot\nabla_\perp\tilde{A}_{0z}^*\big]\,dx\,.\end{aligned}\tag{3.19}$$

This assertion is justified in the two subparagraphs a) and b) below, where the derivative ∂_j with $j\in\{1,2\}$ (but not with $j=3$) is implemented.

a) The first step is to substitute $\partial_j c$ for $\partial_j\tilde{c}$ modulo controlled error terms. From (3.16), we have

$$\partial_j c=\partial_j\tilde{c}+\mathrm{Op}\left(\frac{i\,\xi_j}{|\xi_\perp|}\;\frac{1}{|\xi_\perp|+\langle\xi_\perp\rangle}\right)\tilde{c}\,.\tag{3.20}$$

For $j\in\{1,2\}$, the symbol $|\xi_\perp|^{-1}\,\xi_j$ is bounded. The singularity induced by $\psi_0(\xi_\perp)^{-1}$ is here compensated by the multiplication by ξ_j. It is therefore easy to infer that

$$\begin{aligned}&\left|\int_\Omega a\;\psi_0(D_\perp)\big[b\;\partial_j c\big]\,dx-\int_\Omega a\;\psi_0(D_\perp)\big[b\;\partial_j\tilde{c}\big]\,dx\right|\\&\quad=\left|\int_\Omega \psi_0(D_\perp)a\;b\;\mathrm{Op}\left(\frac{i\,\xi_j}{|\xi_\perp|}\;\frac{1}{|\xi_\perp|+\langle\xi_\perp\rangle}\right)\tilde{c}\,dx\right|\\&\quad\lesssim\parallel b\parallel_{L^\infty}\,\parallel\psi_0(D_\perp)a\parallel_{L^2}\,\parallel\tilde{c}\parallel_{H^{-1}}\lesssim\parallel b\parallel_{L^\infty}\,\parallel a\parallel_{L^2}\,\parallel\tilde{c}\parallel_{L^2}\,.\end{aligned}\tag{3.21}$$

b) The second step is to eliminate the presence of $\psi_0(D_\perp)$ which is displayed in the integral

$$\int_\Omega a\;\psi_0(D_\perp)\big[b\;\partial_j\tilde{c}\big]\,dx\,.$$

Observe that repeated integrations by parts give rise to (for scalar functions a, b and c)

$$\begin{aligned}\int_\Omega a\ \psi_0(D_\perp)\big[b\ \partial_j \tilde c\big]\,dx\ &= -\int_\Omega \partial_j\big[b\ \psi_0(D_\perp)a\big]\ \tilde c\,dx = \int_\Omega a\ b\ \partial_j \tilde c\,dx \\ &\quad + \int_\Omega b\ \tilde c\ \big(1-\psi_0(D_\perp)\big)\partial_j a\,dx + \int_\Omega \partial_j b\ \tilde c\ \big(1-\psi_0(D_\perp)\big)a\,dx\,.\end{aligned}$$

In view of (3.16), the operator $1-\psi_0(D_\perp)$ is of order -2. It follows that

$$\left|\int_\Omega a\ \psi_0(D_\perp)\big[b\ \partial_j \tilde c\big]\,dx - \int_\Omega a\ b\ \partial_j \tilde c\,dx\right| \lesssim \|\, b\,\|_{W^{1,\infty}}\ \|\, a\,\|_{L^2}\ \|\,\tilde c\,\|_{L^2}\,. \tag{3.22}$$

Now, we can come back to the study of (3.19). Due to (3.3) and because $\nabla_\perp\cdot A_{0\perp}=0$, the second line of (3.19) just disappears. Let us consider the first line. Given a vector field $V_\perp = {}^t(V_x, V_y)$, an integration by parts furnishes the identity

$$\begin{aligned}&\int_\Omega \tilde A^*_{0\perp}\cdot V_\perp^\perp\ \nabla_\perp\times\tilde A^*_{0\perp}\,dx \\ &\qquad = \int_\Omega V_y\ (\tilde A^*_{0x}\,\partial_y \tilde A^*_{0x} - \tilde A^*_{0y}\,\partial_x \tilde A^*_{0x})\,dx + \int_\Omega V_x\ (\tilde A^*_{0y}\,\partial_x \tilde A^*_{0y} - \tilde A^*_{0x}\,\partial_y \tilde A^*_{0y})\,dx \\ &\qquad\quad - \int_\Omega \tilde A^*_{0x}\ \tilde A^*_{0y}\ (\partial_x V_y + \partial_y V_x)\,dx + \int_\Omega (\tilde A^{*2}_{0y}\ \partial_x V_x + \tilde A^{*2}_{0x}\ \partial_y V_y)\,dx\,.\end{aligned}$$

Since $\nabla_\perp\cdot\tilde A^*_{0\perp}=0$, applied to the solenoidal vector field $V = v_{0\perp}+A_{0\perp}-A^*_{0\perp}$, the first line gives rise to

$$\frac12\int_\Omega (V_y\ \partial_y|\tilde A_0|^2 + V_x\ \partial_x|\tilde A_0|^2)\,dx = -\frac12\int_\Omega \nabla_\perp\cdot(v_{0\perp}+\tilde A_{0\perp}-\tilde A^*_{0\perp})\ |\tilde A_0|^2\,dx = 0\,.$$

On the other hand, the H^s-bound available on $\tilde U^r_0$ allows to control in the sup norm of one order (horizontal) derivatives of $V = v_{0\perp}+A_{0\perp}-A^*_{0\perp}$ by a constant. Concerning $v_{0\perp}$, this is due to the inclusion $H^s\hookrightarrow Lip$. Concerning $A^*_{0\perp}-A_{0\perp}=\psi_0(D_\perp)\tilde A^*_{0\perp}$, this comes from

$$\|\partial_j(A_{0\perp}-A^*_{0\perp})\|_{L^\infty} \le \|\partial_j\ \psi_0(D_\perp)\tilde A^*_{0\perp}\|_{L^\infty} \le \|\partial_j\ \psi_0(D_\perp)\tilde A^*_{0\perp}\|_{H^{s-1}} \le \|\,\tilde U^r_0\,\|_{H^s}\,.$$

It follows that

$$\left|\int_\Omega \tilde A^*_{0x}\ \tilde A^*_{0y}\ (\partial_x V_y + \partial_y V_x)\,dx - \int_\Omega (\tilde A^{*2}_{0y}\ \partial_x V_x + \tilde A^{*2}_{0x}\ \partial_y V_y)\,dx\right| \lesssim \|\,\tilde U^r_0\,\|_{H^s}\ \|\,\tilde A^*_{0\perp}\,\|^2_{L^2}\,.$$

• Contributions related to **incompressible features**. With again (3.3), this is nothing more than

$$\int_\Omega (\bar{a}\, v_{0\perp} \cdot \nabla_\perp q_1 + \tilde{A}^*_{0\perp} \cdot \psi_0(D_\perp) \nabla_\perp l_1)\, dx$$
$$= - \int_\Omega \bar{a}\, \nabla_\perp \cdot v_{0\perp}\, q_1\, dx - \int_\Omega \psi_0(D_\perp) \nabla_\perp \cdot A^*_{0\perp}\, \psi_0(D_\perp) l_1\, dx = 0\,.$$

• Contributions coming from the **corrector** $v_{1\perp}$. We look here at the influence of

$$\int_\Omega (\bar{a}\, q_0\, \nabla_\perp \cdot v_{1\perp} - \tilde{A}^*_{0\perp} \cdot \psi_0(D_\perp) v^\perp_{1\perp})\, dx = - \int_\Omega \big(\bar{a}\, (\nabla_\perp q_0)^\perp + \psi_0(D_\perp)^2\, A^*_{0\perp}\big) \cdot v^\perp_{1\perp}\, dx\,.$$

Since $\nabla_\perp \cdot A^*_{0\perp} = 0$, the relation (3.7) gives rise to

$$\bar{a}\, (\nabla_\perp q_0)^\perp = - (\mathrm{Id} - \Delta_\perp)^{-1} \big(\nabla_\perp (\nabla_\perp \times A^*_{0\perp})\big)^\perp = (\mathrm{Id} - \Delta_\perp)^{-1}\, \Delta_\perp A^*_{0\perp}\,.$$

On the other hand, the definition (3.16) of ψ_0 means that

$$\psi_0(D_\perp)^2\, A^*_{0\perp} = - (\mathrm{Id} - \Delta_\perp)^{-1}\, \Delta_\perp A^*_{0\perp}\,. \tag{3.23}$$

The sum of the above two expressions is exactly zero.

• Contributions coming from the **corrector** $A^*_{1\perp}$. The matter here is to consider

$$- \int_\Omega v_{0\perp} \cdot (A^{*\perp}_{1\perp} - A^\perp_{1\perp})\, dx + \int_\Omega \tilde{A}^*_{0\perp} \cdot \psi_0(D_\perp)(A^{*\perp}_{1\perp} - A^\perp_{1\perp})\, dx + \int_\Omega \tilde{A}^*_{0z}\, \psi_0(D_\perp)\, \partial_z l_0\, dx\,.$$

Since $\nabla_\perp \cdot v_{0\perp} = 0$ and $\nabla_\perp \cdot A^*_{0\perp} = 0$, we can find a stream function Ψ and some auxiliary scalar function χ such that $v_{0\perp} = \nabla^\perp_\perp \Psi$ and $A^*_{0\perp} = \nabla^\perp_\perp \chi$. With (3.13), it follows that

$$- \int_\Omega v_{0\perp} \cdot (A^*_{1\perp} - A_{1\perp})^\perp\, dx + \int_\Omega \tilde{A}^*_{0\perp} \cdot \psi_0(D_\perp)(A^{*\perp}_{1\perp} - A^\perp_{1\perp})\, dx$$
$$= \int_\Omega \big(\Psi - \psi_0(D_\perp)^2\, \chi\big)\, \nabla_\perp \cdot (A^*_{1\perp} - A_{1\perp})\, dx$$
$$= \int_\Omega \big(\Delta_\perp \Psi - \psi_0(D_\perp)^2\, \Delta_\perp \chi\big)\, \partial_z A_{0z}\, dx\,.$$

Since $\Delta_\perp \Psi = -\nabla_\perp \times v_{0\perp}$ and $\Delta_\perp \chi = -\nabla_\perp \times A^*_{0\perp}$, exploiting (3.6), we find that

$$\int_\Omega \big(\Delta_\perp \Psi - \psi_0(D_\perp)^2\, \Delta_\perp \chi\big)\, \partial_z A_{0z}\, dx + \int_\Omega \tilde{A}^*_{0z}\, \psi_0(D_\perp) \partial_z l_0\, dx$$
$$= \int_\Omega \big(-\nabla_\perp \times v_{0\perp} - \Delta_\perp (\mathrm{Id} - \Delta_\perp)^{-1}\, \nabla_\perp \times A^*_{0\perp}\big)\, (\mathrm{Id} - \Delta_\perp)^{-1}\, \partial_z A^*_{0z}\, dx$$
$$- \int_\Omega \Delta_\perp (\mathrm{Id} - \Delta_\perp)^{-1}\, A^*_{0z}\, \big(\Delta_\perp^{-1}\, \nabla_\perp \times \partial_z v_{0\perp} + (\mathrm{Id} - \Delta_\perp)^{-1}\, \nabla_\perp \times \partial_z A^*_{0\perp}\big)\, dx\,.$$

It suffices to perform one integration by parts with respect to z in the last line to see that it does compensate the preceding line. The right hand side is just zero.

• Contributions involving **extra source terms**. From (3.7), we have

$$\mathrm{a}'(\mathrm{q}) \int_\Omega q_1 \, v_{0\perp} \cdot \nabla_\perp q_0 \, dx = \frac{\mathrm{a}'(\mathrm{q})}{\bar{\mathrm{a}}} \int_\Omega q_1 \, v_{0\perp} \cdot \Psi_0(D_\perp) \Delta_\perp^{-1} \nabla_\perp \nabla_\perp \cdot \, \tilde{A}_{0\perp}^{*\perp} \, dx \, .$$

In view of the second order elliptic equation (3.14), it is clear that q_1 has the same level of regularity than $\tilde{U}_0^r$. Hence

$$\left| \int_\Omega q_1 \, v_{0\perp} \cdot \nabla_\perp q_0 \, dx \right| \lesssim \| \, \tilde{U}_0^r \, \|_{L^\infty} \, \| \, \tilde{U}_0^r \, \|_{L^2}^2 \, .$$

On the other hand, applying the preceding arguments a) and b), we can assert that[2]

$$- \int_\Omega \tilde{A}_{0\perp}^* \cdot \psi_0(D_\perp) \left(v_{0z} \, \nabla_\perp A_{0z}^* \right) dx \tag{3.24}$$

can be replaced (modulo a constant times the square of the L^2-norm of $\tilde{A}_0^*$) by

$$- \int_\Omega \tilde{A}_{0\perp}^* \cdot \left(v_{0z} \, \nabla_\perp \tilde{A}_{0z}^* \right) dx = \int_\Omega \tilde{A}_{0z}^* \, \tilde{A}_{0\perp}^* \cdot \nabla_\perp v_{0z} \, dx$$

where we have used $\nabla_\perp \cdot \tilde{A}_{0\perp}^* = 0$. As a consequence, we have

$$\left| \int_\Omega \tilde{A}_{0\perp}^* \cdot \psi_0(D_\perp) \left(v_{0z} \, \nabla_\perp A_{0z}^* \right) dx \right| \lesssim \| \, \tilde{U}_0^r \, \|_{W^{1,\infty}} \, \| \, \tilde{A}_{0z}^* \, \|_{L^2} \, \| \, \tilde{A}_{0\perp}^* \, \|_{L^2} \, .$$

We turn now to the contribution[3]

$$- \int_\Omega \tilde{A}_{0\perp}^* \cdot \psi_0(D_\perp) \left((A_{0z} - A_{0z}^*) \, \nabla_\perp A_{0z}^* \right) dx \, .$$

Using again a) and b) and the condition $\nabla_\perp \cdot \tilde{A}_{0\perp}^* = 0$, this can be replaced (modulo a constant times the square of the L^2-norm of $\tilde{A}_0^*$) by

[2] In the integral expression below, it must be clear that v_{0z} plays the part of a coefficient which is bounded in H^s with $s > 5/2$, while $\tilde{A}_{0\perp}^*$ and $\tilde{A}_{0z}^*$ are just supposed to be in L^2.

[3] This expression is similar to (3.24) with the H^s-coefficient v_{0z} replaced by the difference $A_{0z} - A_{0z}^*$. However, the H^s-bound on $\tilde{U}_0^r = (q, v, \tilde{A}^*)$ does not furnish directly a Lipschitz control on $A_{0z} - A_{0z}^*$. Some argument is missing here. This is why this contribution is handled separately.

$$-\int_{\Omega} \tilde{A}_{0\perp}^{*} \cdot \big((A_{0z} - A_{0z}^{*})\, \nabla_{\perp} \tilde{A}_{0z}^{*}\big)\, dx = \int_{\Omega} \tilde{A}_{0z}^{*}\, \tilde{A}_{0\perp}^{*} \cdot \nabla_{\perp}(A_{0z} - A_{0z}^{*})\, dx\,.$$

Exploiting (3.17), since $H^{s-1}(\mathbb{R}^3) \hookrightarrow L^{\infty}$ for $s > 5/2$, we find that

$$\| \nabla_{\perp}(A_{0z} - A_{0z}^{*}) \|_{L^{\infty}} \lesssim \| \nabla_{\perp} \psi_0(D_{\perp}) \tilde{A}_{0z}^{*} \|_{H^{s-1}} \lesssim \| \tilde{A}_{0z}^{*} \|_{H^{s}} \lesssim \| \tilde{U}_0^{r} \|_{H^{s}}\ .$$

As a consequence, we can retain that

$$\left| \int_{\Omega} \tilde{A}_{0\perp}^{*} \cdot \psi_0(D_{\perp}) \big((A_{0z} - A_{0z}^{*})\, \nabla_{\perp} A_{0z}^{*}\big)\, dx \right| \lesssim \| \tilde{U}_0^{r} \|_{H^{s}}\ \| \tilde{A}_{0z}^{*} \|_{L^2}\ \| \tilde{A}_{0\perp}^{*} \|_{L^2}\ .$$

In the same vein, using a) to exchange $\nabla_{\perp} A_{0z}^{*}$ with $\nabla_{\perp} \tilde{A}_{0z}^{*}$, and then $\nabla_{\perp} \cdot v_{0\perp} = 0$ to perform the integration by parts, we find that

$$\begin{aligned}
&\left| \int_{\Omega} (A_{0z}^{*} - A_{0z})\, v_{0\perp} \cdot \nabla_{\perp} A_{0z}^{*}\, dx \right| \\
&\qquad \leq \left| \int_{\Omega} (A_{0z}^{*} - A_{0z})\, v_{0\perp} \cdot \nabla_{\perp} \tilde{A}_{0z}^{*}\, dx \right| + \| v_{0\perp} \|_{L^{\infty}}\ \| \tilde{A}_{0z}^{*} \|_{L^2}^{2} \\
&\qquad \leq \left| \int_{\Omega} \tilde{A}_{0z}^{*}\ \nabla_{\perp}(A_{0z}^{*} - A_{0z}) \cdot v_{0\perp}\, dx \right| + \| v_{0\perp} \|_{L^{\infty}}\ \| \tilde{A}_{0z}^{*} \|_{L^2}^{2} \\
&\qquad \lesssim \| \tilde{U}_0^{r} \|_{H^{s}}\ \| \tilde{U}_0^{r} \|_{L^2}^{2}\ .
\end{aligned}$$

And similarly when dealing with

$$\left| \int_{\Omega} \tilde{A}_{0\perp}^{*} \cdot \psi_0(D_{\perp}) \big((A_{0z}^{*} - A_{0z})\, \nabla_{\perp} v_{0z}\big)\, dx \right| \lesssim \| \tilde{U}_0^{r} \|_{H^{s}}\ \| \tilde{U}_0^{r} \|_{L^2}^{2}\ .$$

• Contributions related to the **quasilinear symmetric terms**. We start with the easiest case, which is

$$\bar{\mathrm{a}} \int_{\Omega} q_0\ \partial_z v_{0z}\, dz + \bar{\mathrm{a}} \int_{\Omega} v_{0z}\ \partial_z q_0\, dz = 0\,.$$

Next, we put together

$$\int_{\Omega} v_{0z}\ (A_{0\perp}^{*} - A_{0\perp}) \cdot \nabla_{\perp} A_{0z}^{*}\, dx + \int_{\Omega} \tilde{A}_{0z}^{*}\ \psi_0(D_{\perp}) \big((A_{0\perp}^{*} - A_{0\perp}) \cdot \nabla_{\perp} v_{0z}\big)\, dx\,.$$

We can implement a) and b) to reduce the discussion to

$$\int_\Omega v_{0z}\,(A^*_{0\perp}-A_{0\perp})\cdot\nabla_\perp\tilde{A}^*_{0z}\,dx+\int_\Omega \tilde{A}^*_{0z}\,\big((A^*_{0\perp}-A_{0\perp})\cdot\nabla_\perp v_{0z}\big)\,dx=0\,.$$

We have also to consider

$$\int_\Omega v_{0\perp}\cdot(A^*_{0\perp}-A_{0\perp})^\perp\,(\nabla_\perp\times A^*_{0\perp})\,dx+\int_\Omega \tilde{A}^*_{0\perp}\cdot\psi_0(D_\perp)\big((A^*_{0\perp}-A_{0\perp})^\perp\,(\nabla_\perp\times v_{0\perp})\big)\,dx\,.$$

Again, thanks to a) and b), it suffices to look at

$$\int_\Omega v_{0\perp}\cdot(A^*_{0\perp}-A_{0\perp})^\perp\,(\nabla_\perp\times\tilde{A}^*_{0\perp})\,dx+\int_\Omega \tilde{A}^*_{0\perp}\cdot(A^*_{0\perp}-A_{0\perp})^\perp\,(\nabla_\perp\times v_{0\perp})\,dx\,.$$

But the action of $\nabla_\perp\times$ is self-adjoint. This expression can be bounded by

$$\|A^*_{0\perp}-A_{0\perp}\|_{W^{1,\infty}}\ \|\tilde{U}^r_0\|^2_{L^2}\,.$$

• Contributions implying **the divergence of** $v_{0\perp}$. Since $\nabla_\perp\cdot v_{0\perp}=0$, they can be ignored. The same applies indirectly to

$$\frac{1}{2}\int_\Omega v_{0\perp}\cdot\nabla_\perp\big(|A^*_0-A_0|^2\big)\,dx=-\frac{1}{2}\int_\Omega \nabla_\perp\cdot v_{0\perp}\,|A^*_0-A_0|^2\,dx=0\,.$$

Notice that a positive bulk viscosity ($\nu>0$) is not required. The Cauchy problem for the modulation equations is well-posed even if $\nu=0$.

Summary. In conclusion, we have proved that

$$\frac{d}{dt}\,\|\tilde{U}^r_0\|^2_{L^2}\lesssim\|\tilde{U}^r_0\|_{H^s}\,\|\tilde{U}^r_0\|^2_{L^2}\,.$$

By Grönwall's inequality, the L^2-norm of $\tilde{U}^r_0$ remains under control. This is the key element for the proof of Proposition 4.

4. Asymptotic analysis on a toy model

As a first step, we propose to consider a (linear) simplified model. This helps to better delineate the basic ingredients of the proof (of Theorem 1) and its underlying difficulties. Instead of looking directly with (1.1)-(1.2)-(1.3), in this section, we work with its reduced version (1.2)-(4.1)-(4.2) elucidated below. In Subsection 4.1, we will perform a Fourier analysis which reveals the importance of using an adequate (matrix-valued) symmetrizer. This symmetrizer takes the form of a Fourier multiplier whose properties (including a lack of smoothness) are examined in Subsection 4.2. To avoid the presence of singularities, special conditions are needed. In return, a complete WKB analysis becomes available for our linear model, as it is detailed in Subsection 4.3.

From (1.1), we isolate the penalized terms, and we freeze the corresponding coefficients. The only exception is the term $\varepsilon^{-1}(A^*-A)\times e_z$ appearing in the last line, which is temporarily omitted from the analysis and which comes from the Hall effect. In fact the coefficient $d = d_i/d_e$ should appear in that factor but it is invisible here since it has been normalized ($d = 1$). Its impact will be discussed in a later section. Let us just consider

$$\begin{cases} \partial_t q + \dfrac{1}{\varepsilon}\, \nabla_\varepsilon \cdot v = 0\,, \\ \partial_t v + \dfrac{1}{\varepsilon}\, \nabla_\varepsilon q - \dfrac{1}{\varepsilon}\, (A^* - A) \times e_z = 0\,, \\ \partial_t A^* - \dfrac{1}{\varepsilon}\, v \times e_z + \dfrac{1}{\varepsilon}\, \nabla_\varepsilon l = 0\,. \end{cases} \tag{4.1}$$

We keep (1.2), and we simply replace (1.3) by

$$\frac{1}{\varepsilon}\, (A^* - A + \Delta_\varepsilon A) = 0\,. \tag{4.2}$$

The system (4.1), which is linear, is amenable to a Fourier analysis. This facilitates an investigation of the behavior of its solutions, given a fixed frequency $\xi \in \mathbb{R}^3$.

4.1. The Fourier side

We adopt the convention

$$\hat{U}^r(t,\xi) = (\hat{q}, \hat{v}, \hat{A}^*)(t,\xi) := \int_{\mathbb{R}^3} e^{-i\,x\cdot\xi}\, U^r(t,x)\, dx\,, \qquad U^r = (q, v, A^*)\,.$$

We look at $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ as a fixed parameter. We define $\xi_\varepsilon := {}^t(\xi_1, \xi_2, \varepsilon\, \xi_3)$, and we put aside $\xi_\perp := {}^t(\xi_1, \xi_2)$. We have to handle the following linear system of ODEs

$$\begin{cases} \partial_t \hat{q} + \dfrac{i}{\varepsilon}\, \xi_\varepsilon \cdot \hat{v} = 0\,, \\ \partial_t \hat{v} + \dfrac{i}{\varepsilon}\, \xi_\varepsilon\, \hat{q} - \dfrac{1}{\varepsilon}\, (\hat{A}^* - \hat{A}) \times e_z = 0\,, \\ \partial_t \hat{A}^* - \dfrac{1}{\varepsilon}\, \hat{v} \times e_z + \dfrac{i}{\varepsilon}\, \xi_\varepsilon\, \hat{l} = 0\,. \end{cases} \tag{4.3}$$

The divergence free condition (1.2) gives rise to the polarization condition

$$\frac{1}{\varepsilon}\, \xi_\varepsilon \cdot \hat{A}^* = 0\,, \tag{4.4}$$

while the relation (1.3) leads to

$$\frac{1}{\varepsilon}\, (\hat{A}^* - \hat{A} - |\xi_\varepsilon|^2\, \hat{A}) = 0\,. \tag{4.5}$$

The initial data (1.5) together with (1.7) yield

$$\hat{U}^r_\varepsilon(0,x) = \hat{U}^{r\,in}_\varepsilon(x) = (\hat{q}^{in}_\varepsilon, \hat{v}^{in}_\varepsilon, \hat{A}^{in*}_\varepsilon)(x)\,, \qquad \xi_\varepsilon \cdot \hat{A}^{in*}_\varepsilon = 0\,. \tag{4.6}$$

In Paragraph 4.1.1, we highlight a challenge that arises when analyzing (4.3), and by extension (4.1). The system (4.3) is not directly exploitable on its own. However, by incorporating the relations (4.4) and (4.5), we can eliminate the redundant state variables $\hat{A}$ and $\hat{l}$. This is the objective of Paragraph 4.1.2.

4.1.1. First hurdle

When $\xi_\varepsilon = 0$, (4.4) disappears; (4.5) furnishes $\hat{A}^* = \hat{A}$; on the other hand, (4.3) reduces to

$$\begin{cases} \partial_t \hat{q} = 0\,, \\ \partial_t \hat{v} = 0\,, \\ \partial_t \hat{A}^* - \dfrac{1}{\varepsilon}\, \hat{v} \times e_z = 0\,. \end{cases} \tag{4.7}$$

Starting from (4.6), the solution is given by

$$\hat{q}(t,0) = \hat{q}^{in}(0), \qquad \hat{v}(t,0) = \hat{v}^{in}(0), \qquad \hat{A}^*(t,0) = A^{in*}(0) + \frac{t}{\varepsilon}\, \hat{v}^{in}_\perp(0)\,. \tag{4.8}$$

Even for prepared data $v^{in}(0,\cdot)$, satisfying $\xi_1\, \hat{v}^{in}_x(0,\xi) + \xi_2\, \hat{v}^{in}_y(0,\xi) = 0$, it is always possible to adjust $v^{in}(0)$ in such a way that $\hat{v}^{in}_\perp(0) \not\equiv 0$. It suffices for instance to impose $\hat{v}^{in}(0,\xi) = {}^t({}^t\xi^\perp_\perp, 0)$ for $|\xi| \leq 1$. This induces a linear growth of the L^2-norm of $\hat{U}^r(t,\cdot)$ with respect to t. This indicates that the system (4.3) - or more generally a derived system implying $\hat{U}^r$ - cannot be put in a symmetric form (with purely imaginary eigenvalues) because, in this case, the solutions would stay bounded.

For $\xi_\varepsilon = 0$, we observe from (4.7) that 0 is an eigenvalue of maximal multiplicity 7, which is accompanied by a non-zero nilpotent term. Consequently, for $\xi_\varepsilon = 0$, the control of $\hat{U}^r(\varepsilon,t,\cdot)$ in terms of the initial data $\hat{U}^r(\varepsilon,0,\cdot)$ leads to a loss of (at least) one negative power of ε. This instability points to the necessity of a singular change of state variables to restore uniform estimates.

Prior to that, we have to better describe the locus of points $(\varepsilon,\xi) \in \mathbb{R}^*_+ \times \mathbb{R}^3$ satisfying $\xi_\varepsilon = 0$. In fact, the vector ξ_ε is built from ε and ξ through the smooth map

$$\begin{array}{rcl} \mathcal{T} : \mathbb{R}_+ \times \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (\varepsilon,\xi) & \longmapsto & \xi_\varepsilon\,. \end{array}$$

We have just highlighted the values of ε and ξ which are adjusted such that $\xi_\varepsilon = 0$, that is which belong to

$$Sing := \mathcal{T}^{-1}\big(\{0\}\big) = Sing_1 \cup Sing_2, \quad Sing_1 := \mathbb{R}^*_+ \times \{0\}, \quad Sing_2 := \{0\} \times D_v,$$

where $D_v \subset \mathbb{R}^3$ is the vertical line of equation

$$D_v := \{\xi \in \mathbb{R}^3\,;\, \xi_\perp = 0\} = \big\{(0,0,\xi_3);\, \xi_3 \in \mathbb{R}\big\}. \tag{4.9}$$

In practice, we will work with $\varepsilon \in \mathbb{R}^*_+$. But, due to the presence of the set $Sing_1$, at low frequencies $|\xi| \sim \varepsilon$, there is a loss of uniform estimates with respect to ε. Moreover, this amplification may occur near $Sing_2$, when $\xi_\perp \simeq 0$ (even if ξ_3 stays away from 0). In what follows, we must take care of what happens near $Sing$.

4.1.2. An important step towards simplification

For $\xi_\varepsilon \neq 0$, we denote by $P \equiv P_\varepsilon \equiv P(\varepsilon, \xi)$ the orthogonal projector from $\mathbb{R}^3$ onto the plane which is orthogonal to the direction ξ_ε. In other words

$$P\,\hat{A}^* = -\,|\xi_\varepsilon|^{-2}\,\xi_\varepsilon \times (\xi_\varepsilon \times \hat{A}^*)\,. \tag{4.10}$$

The constraint (4.4) is equivalent to

$$P(\varepsilon, \xi)\,\hat{A}^* = \hat{A}^*\,. \tag{4.11}$$

Introduce the scalar function $\psi \equiv \psi_\varepsilon(\xi) \equiv \psi(\varepsilon, \xi)$ which is defined by

$$\psi = \Psi \circ \mathcal{T}(\varepsilon, \xi) = \Psi(\xi_\varepsilon)\,, \qquad \Psi(\xi) := \frac{|\xi|}{\langle \xi \rangle} = \frac{(\xi_1^2 + \xi_2^2 + \xi_3^2)^{1/2}}{(1 + \xi_1^2 + \xi_2^2 + \xi_3^2)^{1/2}}\,. \tag{4.12}$$

Exploiting (4.5), we can express $\hat{A}$ in terms of $\hat{A}^*$. By this way, we can eliminate from (4.3) the presence of $\hat{A}$ through the relation

$$\hat{A}^* - \hat{A} = \psi^2\,\hat{A}^*. \tag{4.13}$$

On the other hand, taking into account (4.11) or (4.4), the third equation of (4.3) gives rise to

$$\hat{l} = -\frac{i}{|\xi_\varepsilon|^2}\,\xi_\varepsilon \cdot (\hat{v} \times e_z).$$

We see here that the polarization condition (4.11) or (4.4) is allowed by the presence of the Lagrange multiplier l. For $\xi_\varepsilon \neq 0$, we can deduce $\hat{l}$ from $\hat{v}$. But, for $\xi_\varepsilon = 0$, without special assumptions on $\hat{v}$, the right hand side is not well defined. This observation underscores again the importance of treating the set $Sing$ as a special case.

After elimination of $\hat{A}$ and $\hat{l}$, we recover a system of 6 equations on 6 unknowns which are: $\hat{q}$ (a scalar), $\hat{v}$ (a vector in $\mathbb{R}^3$) and $P\hat{A}^*$ (which for $\xi_\varepsilon \neq 0$ must be orthogonal to ξ_ε, and therefore implies only two independent components). Knowing (4.11) and (4.13), there remains to solve

$$\begin{cases} \partial_t \hat{q} + \dfrac{i}{\varepsilon}\,\xi_\varepsilon \cdot \hat{v} = 0\,, \\[2mm] \partial_t \hat{v} + \dfrac{i}{\varepsilon}\,\xi_\varepsilon\,\hat{q} - \dfrac{1}{\varepsilon}\,\psi^2\,(P\,\hat{A}^*) \times e_z = 0\,, \\[2mm] \partial_t P\hat{A}^* - \dfrac{1}{\varepsilon}\,P\,(\hat{v} \times e_z) = 0\,. \end{cases} \tag{4.14}$$

The function $\psi(\varepsilon, \cdot)$ is associated with a scalar Fourier multiplier. Let us introduce

$$\tilde{A}^* := \psi(\varepsilon, D)\,A^*\,, \qquad \mathcal{F}\tilde{A}^* = \psi(\varepsilon, \xi)\,\hat{A}^*\,. \tag{4.15}$$

This means to introduce the new unknown

$$\mathcal{V} \equiv \mathcal{V}(\varepsilon, t, \xi) := (q, v, \tilde{A}^*) . \tag{4.16}$$

Remark 5 *(Comparison with the change (3.15)). For $\varepsilon = 0$, we find $\psi(0, \xi) = \psi_0(\xi_\perp)$, and the operation (4.15) simplifies to (3.15). The introduction of $\tilde{A}^*$ extends to the full system what has been done concerning the modulation equations.*

The matrix $P(\varepsilon, \cdot)$ is associated with a matricial Fourier multiplier. Assuming (4.11), the definition (4.16) is the same as

$$\tilde{A}^* := \psi(\varepsilon, D)\, P(\varepsilon, D)\, A^* , \qquad \mathcal{F}\tilde{A}^* = \psi(\varepsilon, \xi)\, P(\varepsilon, \xi)\, \hat{A}^* . \tag{4.17}$$

Let us multiply the last line of (4.14) by ψ in order to obtain

$$\partial_t \hat{\mathcal{V}} + \frac{1}{\varepsilon}\, \mathcal{A}_\varepsilon\, \hat{\mathcal{V}} = 0 , \tag{4.18}$$

where $\mathcal{A} \equiv \mathcal{A}_\varepsilon \equiv \mathcal{A}(\varepsilon, \xi)$ is the square matrix of size 7×7 defined by

$$\mathcal{A}_\varepsilon := \begin{pmatrix} 0 & i\, {}^t\xi_\varepsilon & 0 \\ i\, \xi_\varepsilon & 0 & \psi_\varepsilon\, J\, P_\varepsilon \\ 0 & \psi_\varepsilon\ P_\varepsilon\ J & 0 \end{pmatrix} , \qquad J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -{}^t J . \tag{4.19}$$

By construction, at time $t = 0$, we must impose

$$\hat{\mathcal{V}}(\varepsilon, 0, x) = (\hat{q}^{in}, \hat{v}^{in}, \mathcal{F}\tilde{A}^{in*})(\varepsilon, x) , \qquad \tilde{A}^{in*} := \psi(\varepsilon, D)\, A^{in*}. \tag{4.20}$$

Remark 6 *(About the polarization conditions). From (4.6), we have $\tilde{A}^{in*} = P_\varepsilon(D)\, \tilde{A}^{in*}$. On the other hand, from (4.18), we get that $\partial_t \big(\mathrm{Id} - P_\varepsilon(D)\big)\tilde{A}^* = 0$. It follows that the constraint $\tilde{A}^* = P_\varepsilon(D)\, \tilde{A}^*$ is propagated by (4.18). Using (4.15) or (4.17) is the same in the context of our toy model. Assuming (1.7), when dealing with (4.18), we can forget the polarization conditions. But retain that the projector P_ε enters in the definition of $\mathcal{A}_\varepsilon$.*

The matrix $\mathcal{A}_\varepsilon : \mathbb{C}^7 \to \mathbb{C}^7$ is skew-selfadjoint (with $\mathcal{A}_\varepsilon^\dagger := {}^t\bar{\mathcal{A}}_\varepsilon$, we have $\mathcal{A}_\varepsilon^\dagger = -\mathcal{A}_\varepsilon$). Its spectrum is purely imaginary. Thus, the equation (4.18) generates a unitary semi-group for the hermitian norm. In particular, its solutions remain bounded for all times, for all values of ξ_ε, including $\xi_\varepsilon = 0$. This differs from what has been observed at the level of (4.8). This is because the multiplication of the last line of (4.14) by ψ, which takes the value 0 for $\xi_\varepsilon = 0$, has the effect of erasing the singularity at the position $\xi_\varepsilon = 0$.

By construction, the expression $\mathcal{V} := (q, v, \tilde{A}^*)$ is subject to the Cauchy problem

$$\partial_t \mathcal{V} + \frac{1}{\varepsilon}\, \mathcal{A}_\varepsilon(D)\, \mathcal{V} = 0 , \qquad \mathcal{V}(0, \cdot) = \mathcal{V}^{in} . \tag{4.21}$$

The system (4.21) is self-contained. It implies the action of $\mathcal{A}_\varepsilon(D)$ which is skew-adjoint. It is therefore more suitable than (4.1) in the perspective of a complete asymptotic analysis.

4.2. Implementation of the basic tools

In Paragraphs 4.2.1 and 4.2.2, we investigate the behavior of the functions ψ and P which are crucial components in the analysis. In Paragraph 4.2.3, we evaluate the merits and drawbacks of a formal calculus whether it is led from (4.1) or (4.18).

4.2.1. Properties of ψ

The function ψ, as defined by (4.12), appears as the composition of Ψ and $\mathcal{T}$. The map $\mathcal{T}$ is smooth on its domain of definition but the function Ψ is not. Indeed, Ψ is continuous on $\mathbb{R}^3$ without being continuously differentable at the origin. It follows that ψ involves singularities which are localized along $Sing = \mathcal{T}^{-1}(\{0\})$.

To derive from (4.14) asymptotic expansions in powers of ε, we fix $\xi \in \mathbb{R}^3$ and we let $\varepsilon \in \mathbb{R}_+^*$ go to 0. The case $\xi = 0$ has already been addressed in Paragraph 4.1.1. We can suppose that $\xi \neq 0$. By this way, we avoid $Sing_1$. Still, we are faced with $Sing_2$. With this in mind, the positions of ξ which are inside D_v and outside D_v must be distinguished.

- For $\xi_\perp \neq 0$, there exists a Taylor expansion at all order N, namely

$$\psi(\varepsilon, \xi) = \psi_0(\xi) + \sum_{n=2}^{N} \frac{\varepsilon^n}{n!} (\partial_\varepsilon^n \psi)(0, \xi) + O(\varepsilon^{N+1}), \qquad N \geq 2. \tag{4.22}$$

  In particular, we can compute

$$\psi_0(\xi) \equiv \psi_0(\xi_\perp) = \frac{|\xi_\perp|}{\langle \xi_\perp \rangle}, \qquad (\partial_\varepsilon^2 \psi)(0, \xi) = \frac{\xi_3^2}{\langle \xi_\perp \rangle} \Big(\frac{1}{|\xi_\perp|} - \frac{|\xi_\perp|}{\langle \xi_\perp \rangle^2} \Big).$$

- For $\xi_\perp = 0$, we directly find that

$$\psi(\varepsilon, 0, 0, \xi_3) = \frac{\varepsilon\, |\xi_3|}{(1 + \varepsilon^2 \xi_3^2)^{1/2}} = \sum_{n=1}^{N} c_n(\xi_3)\, \varepsilon^n + O(\varepsilon^{N+1}), \quad c_1(\xi_3) := |\xi_3|. \tag{4.23}$$

  There is still an expansion in powers of ε. However, the coefficients c_j are not smooth near $\xi_3 = 0$, as it is the case concerning c_1.

There is a significant difference between (4.22) and (4.23). For $\xi_3 \neq 0$, the coefficient in factor of ε inside (4.22) is just 0 for all $\xi_\perp \neq 0$, whereas it is equal to $|\xi_3| \neq 0$ at the level of (4.23). This jump along D_v implies that the expansions (4.22) and (4.23) cannot be connected continuously in ξ, particularly when $\xi_\perp$ goes to zero while $\xi_3 \neq 0$ is fixed. Following the same logic (presence of a singularity along D_v), we can observe that the derivatives $\partial_\xi^\alpha \psi(0, \xi)$ with $|\alpha| \geq 2$ exhibit explosive behavior (for $|\xi_3| \neq 0$) when $|\xi_\perp|$ goes to 0. As a consequence, they cannot be continuously extended up to $\xi_\perp = 0$.

For $\varepsilon \in]0, 1]$, remark that

$$0 < \varepsilon\, \langle \xi \rangle^{-1} |\xi| \leq \psi(\varepsilon, \xi) \leq 1, \qquad \forall \xi \in \mathbb{R}^3.$$

The function $\psi(\varepsilon,\cdot)$ is therefore associated with a pseudo-differential symbol which (for all fixed $\varepsilon \in \mathbb{R}_+^*$) is elliptic of order 0 on $\mathbb{R}^3 \setminus \{0\}$. More precisely, for large frequencies $|\xi_\varepsilon| \gg 1$, we find that $\psi_\varepsilon \sim 1$, and it is easy to invert ψ uniformly with respect to ε. However, the symbol $\psi(\varepsilon,\cdot)$ degenerates near $\xi = 0$ since $\psi(\varepsilon, 0) = 0$. Thus, for low frequencies $|\xi_\varepsilon| \ll 1$, the function $\psi(\varepsilon,\cdot)$ moves towards 0. And there is no control on the ellipticity which could be uniform with respect to ε. In other words, there is no function ψ_1 which could be adjusted in such a way that

$$0 < \psi_1(\xi) \le \psi(\varepsilon,\xi), \qquad \forall(\varepsilon,\xi) \in]0,1] \times (\mathbb{R}^3 \setminus \{0\}).$$

This makes a return (without loss in ε) from $\mathcal{F}\tilde{A}^*$ to $\hat{A}^*$ problematic since we pass from $\mathcal{F}\tilde{A}$ to $\hat{A}^*$ by dividing by ψ while, for $\xi_\perp = 0$ and $|\xi_3| \neq 0$, the function $\psi(\cdot, 0, 0, \xi_3)^{-1}$ is not bounded (with respect to $\varepsilon \in]0,1]$). Similarly, the reconstitution of U from $\mathcal{V}$ implements the operator $\psi(\varepsilon, D)^{-1}$. There is consequently a loss by a factor ε^{-1}. The instability detected at the level of (4.8) is in fact lifted by resorting to ψ.

Remark 7 *(The link between B and $\tilde{A}^*$). In the context of our toy model, the magnetic field B_ε can be expressed in terms of $\tilde{A}^*$ through*

$$\hat{B}_\varepsilon = i\,\xi_\varepsilon \times \hat{A}_\varepsilon = \frac{i}{\langle \xi_\varepsilon \rangle}\,\frac{\xi_\varepsilon}{|\xi_\varepsilon|} \times \mathcal{F}\tilde{A}^*\,.$$

This means that the uniform L^2-control on $\tilde{A}^*$ (that will be obtained in the proof) implies a uniform H^1-bound on the magnetic field B.

4.2.2. Properties of P

As long as $\xi \neq 0$ is fixed, the function $P(\cdot,\xi)$ can be extended by continuity up to $\varepsilon = 0$ through

$$P_0\,u = P_0(\xi_\perp)\,u := \begin{cases} -|\xi_\perp|^{-2}\,{}^t({}^t\xi_\perp,0) \times \big({}^t({}^t\xi_\perp,0)\times u\big) & \text{if } \xi_\perp \neq 0\,, \\ {}^t(u_1,u_2,0) & \text{if } \xi_\perp = 0\,. \end{cases} \tag{4.24}$$

Looking at P_0 (which is not continuous along $\xi_\perp = 0$), we recover the dichotomy already observed between the line D_v and its complementary set. That being said, for all $\xi \neq 0$, the function $[0,1] \ni \varepsilon \mapsto P(\varepsilon,\xi)$ is of class C^∞. Indeed:

- For $\xi_\perp \neq 0$, this can be seen at the level of (4.10), and we have

$$P(\varepsilon,\xi) = \sum_{j=0}^{N} \varepsilon^j\,P_j(\xi) + O(\varepsilon^{N+1})\,. \tag{4.25}$$

  In particular, we can compute

$$P_1(\xi) = -|\xi_\perp|^{-2}\,\xi_3\,\big(e_z \times {}^t({}^t\xi_\perp,0) + {}^t({}^t\xi_\perp,0)\times e_z\big)\times\ . \tag{4.26}$$

- For $\xi_\perp = 0$ and $\xi_3 \neq 0$, the map $P(\varepsilon,0,0,\xi_3)$ is simply constant, equal to the second line of (4.24). In other words

$$P(\varepsilon,0,0,\xi_3)=P_0(0)\,,\qquad \forall\varepsilon\in\mathbb{R}_+^*\,. \tag{4.27}$$

The map $P(\cdot,\cdot)$ is continuous on $]0,1]\times(\mathbb{R}^3\setminus\{0\})$ but it is not on $[0,1]\times(\mathbb{R}^3\setminus\{0\})$. Indeed, for $\xi={}^t({}^t\xi_\perp,\xi_3)$ with $\xi_\perp\neq 0$ and $\xi_3\neq 0$, we have

$$\lim_{n\to+\infty}\;P\big(n^{-1},{}^t(n^{-1}\,{}^t\xi_\perp,\xi_3)\big)=P(1,\xi)\neq P_0\big(0,{}^t(0,\xi_3)\big)\,.$$

Again, for all $\xi\neq 0$, the expression $P(\cdot,\xi)$ can be expanded with respect to $\varepsilon=0$ with a precision valid at all orders. But the corresponding expansions do not coincide when $|\xi_\perp|$ goes to 0 with $\xi_3\neq 0$.

4.2.3. Two alternative methods to perform the WKB calculus

In the perspective of the asymptotic analysis, we can start from (4.1), (4.3), (4.14) or (4.18). As pointed out above, these approaches are not all equivalent from the viewpoint of uniform estimates with respect to the parameter $\varepsilon\in]0,1]$.

A first strategy would be to argue directly from (4.1). After Fourier transformation, it is the same thing as working with (4.3) or (4.14). Since (4.3) involves through (4.13) the expression ψ^2, and therefore Ψ^2 which is smooth with respect to ε for all ξ, there is no difficulty in extracting a formal cascade of WKB equations. Besides, this is what has been done (up to the order one) in Section 3. This process sounds straightforward and adaptable to nonlinear situations. But it is misleading: it can really provide false information when going up to $N\geq 2$ or when performing energy estimates.

Indeed, it is unrealistic to obtain and justify a complete asymptotic calculus by starting from a system which is not well-posed uniformly with respect to ε. This is precisely the case of (4.3) in view of (4.8). There is no guarantee that the cascade of WKB equations thus obtained could be solved and justified for all N. In this regard, the situation $N=1$ is quite apart from $N\geq 2$, precisely because the singularities of ψ are not yet engaged. It is important to realize that passing through (4.1), (4.3) or (4.14) is not suitable even to prove that the modulation equations are meaningful.

The second strategy is to consider (4.18) or (4.21). This means to seek approximate solutions $\hat{\mathcal{V}}^a_\varepsilon$ in the form

$$\hat{\mathcal{V}}^a_\varepsilon(t,\cdot)=\hat{\mathcal{V}}_0(t,\xi)+\varepsilon\,\hat{\mathcal{V}}_1(t,\xi)+\varepsilon^2\,\hat{\mathcal{V}}_2(t,\xi)+\cdots+\varepsilon^N\,\hat{\mathcal{V}}_N(t,\xi)\,, \tag{4.28}$$

as well as corresponding exact solutions $\mathcal{V}_\varepsilon$. As will be seen in the next subsection, this is compatible with uniform H^s-energy estimates which are crucial for the construction of families of solutions (with a uniform lifespan) and for stability. Of course, the comeback from $\mathcal{V}_\varepsilon$ to U_ε is a bit delicate near the singularity. But this is a minor problem.

4.3. Asymptotic analysis on a symmetrized version of the toy model

We fix $\xi\neq 0$, and we focus on (4.18). The framework induced by (4.18) is quire classical in non linear geometric optics [25,27]. However, there are all kinds of subtleties, especially those related to the presence of D_v, see the unusual condition (4.42). For the sake of completeness, we give in this subsection a few details. In Paragraph 4.3.1, we describe the formal calculus. In paragraph 4.3.2, we proceed with its justification. In Paragraph 4.3.3, we explain how to come back to the initial state variable U_ε.

4.3.1. *The formal calculus*

We begin by highlighting a notion of approximate solution which is similar to Definition 2, while being more adapted to the formulation (4.18). It is especially necessary to take into account the presence of data constrained by (4.11).

Definition 8. *Let us fix some integer $N \in \mathbb{N}$ and some time $T \in \mathbb{R}_+^*$. We say that the expression $\hat{\mathcal{V}}^a(\varepsilon, t, \xi) = (\hat{q}^a, \hat{v}^a, \mathcal{F}\tilde{A}^{*a})(\varepsilon, t, \xi)$ is an approximate solution to (4.18) which is of order N on the interval $[0, T]$ when there exists a constant $C \in \mathbb{R}_+^*$ such that*

$$\partial_t \hat{\mathcal{V}}_\varepsilon^a + \frac{1}{\varepsilon} \mathcal{A}_\varepsilon \hat{\mathcal{V}}_\varepsilon^a = \hat{R}_\varepsilon^a = (\hat{R}_{q_\varepsilon}^a, \hat{R}_{v_\varepsilon}^a, \hat{R}_{\tilde{A}_\varepsilon^*}^a), \qquad \| \hat{R}_\varepsilon^a \|_{L^\infty([0,T];L^2(\mathbb{R}^3))} \le C \varepsilon^N, \tag{4.29}$$

as well as

$$\| (\mathrm{Id} - P_\varepsilon) \mathcal{F}\tilde{A}^{*a}(\varepsilon, 0, \cdot) \|_{L^2(\mathbb{R}^3)} \le C \varepsilon^N. \tag{4.30}$$

A few remarks are in order. We do not specify the content of $\hat{\mathcal{V}}_\varepsilon^a$ which can be given by an expansion like $\hat{\mathcal{V}}_0 + \varepsilon \hat{\mathcal{V}}_1 + \cdots$ or not. The integer N may be large or small ($N = 1$ is allowed). The condition (4.30) implies data which must be sufficiently prepared. Indeed, in practice, it could be difficult to guarantee (4.11) at time $t = 0$. A margin of error as in (4.30) is welcome.

Since the map $[0, 1] \ni \varepsilon \mapsto \mathcal{A}_\varepsilon(\xi)$ is of class C^∞, we get access near $\varepsilon = 0$ to a complete expansion of $\mathcal{A}_\varepsilon(\xi)$ in powers of ε, that is

$$\mathcal{A}_\varepsilon = \mathcal{A}_0 + \varepsilon \mathcal{A}_1 + \varepsilon^2 \mathcal{A}_2 + \cdots .$$

In particular

$$\mathcal{A}_0 = \mathcal{A}(0, \xi) \equiv \mathcal{A}_0(\xi_\perp) = \begin{pmatrix} 0 & i\,({}^t\xi_\perp, 0) & 0 \\ i\,{}^t({}^t\xi_\perp, 0) & 0 & \psi_0(\xi_\perp)\, J\, P_0(\xi_\perp) \\ 0 & \psi_0(\xi_\perp)\, P_0(\xi_\perp)\, J & 0 \end{pmatrix}.$$

On the other hand

$$\mathcal{A}_1 = \partial_\varepsilon \mathcal{A}(0, \xi) = \begin{pmatrix} 0 & i\,\xi_3\, {}^t e_z & 0 \\ i\,\xi_3\, e_z & 0 & \psi_0\, J\, P_1(\xi) \\ 0 & \psi_0\, P_1(\xi)\, J & 0 \end{pmatrix}. \tag{4.31}$$

We plug $\hat{\mathcal{V}}_\varepsilon^a(t, \cdot)$ into (4.18), and $\hat{\mathcal{V}}_\varepsilon^a(0, \cdot)$ into (4.11). By this way, we obtain asymptotic series with respect to increasing powers of ε. Below, the two cases $\xi_\perp = 0$ and $\xi_\perp \neq 0$ are dealt with separately (this is possible because the situation is here linear).

• For $\xi_\perp = 0$ (and $\xi_3 \neq 0$), we see that $\mathcal{A}_0(0) = 0$. There is no penalized term. We put $\hat{\mathcal{V}}_\varepsilon^a$ as in (4.28) inside (4.18) to extract

$$\partial_t \hat{\mathcal{V}}_0 + \mathcal{A}_1(0, 0, \xi_3)\, \hat{\mathcal{V}}_0 = 0 .$$

Then, knowing the content of the $\hat{\mathcal{V}}_k$ for $k \le j - 1$, we must impose

$$\partial_t \hat{\mathcal{V}}_j + \mathcal{A}_1(0,0,\xi_3)\,\hat{\mathcal{V}}_j + \sum_{k=1}^{j} \mathcal{A}_{k+1}(0,0,\xi_3)\,\hat{\mathcal{V}}_{j-k} = 0\,. \tag{4.32}$$

Furthermore, in view of (4.27), the condition (4.11) interpreted in terms of (4.30) leads to

$$\mathcal{F}\tilde{A}^*_{zj}(0,0,0,\xi_3) = 0\,, \qquad \forall j \in \{0,\cdots,N\}\,. \tag{4.33}$$

This sequence of equations can easily be solved step by step. Moreover, since the matrix $\mathcal{A}_1$ is skew-adjoint, the norm of $\hat{\mathcal{V}}_0$ is conserved

$$\| \hat{\mathcal{V}}_0(t,0,0,\xi_3) \| \leq \| \hat{\mathcal{V}}_0(0,0,0,\xi_3) \|\,, \qquad \forall t \in \mathbb{R}\,.$$

The access to $\hat{\mathcal{V}}_j(t,0,0,\xi_3)$ is via (4.32) which implies products as source terms, for instance $\mathcal{A}_{j+1}(0,0,\xi_3)\,\mathcal{V}_0$. In this way, the growth of the $\mathcal{A}_j(0,0,\xi_3)$ with respect to ξ_3 can be transmitted to the $\hat{\mathcal{V}}_j(t,0,0,\xi_3)$, and this may compromise L^2-controls. To remedy this, the initial data $\hat{\mathcal{V}}_j(0,0,0,\xi_3)$ are usually chosen rapidly decreasing with respect to ξ_3. We can specify a decreasing rate like $|\xi_3|^{-k}$ for all $k \in \mathbb{N}$. Or, given a desired precision ε^{-N} with N fixed, we can work with a well-adjusted finite decreasing rate like $|\xi_3|^{-k}$ with k large enough. Expressed in terms of functions depending on x, the easiest set-up is to involve data in H^∞.

• Let us turn now to the second case $\xi_\perp \neq 0$. This time, the formal calculus furnishes a non-zero term with ε^{-1} in factor, which is $\mathcal{A}_0\,\hat{\mathcal{V}}_0 = 0$. The matrix $\mathcal{A}_0$ is skew-adjoint. We denote by $\mathcal{P}_0$ the orthogonal projector onto its kernel, and by $\mathcal{Q}_0 := \mathrm{Id} - \mathcal{P}_0$. We do not want to trigger oscillations. With this in mind, we have to prepare $\hat{\mathcal{V}}_0$ in the kernel of $\mathcal{A}_0$, which amounts to impose $\hat{\mathcal{V}}_0 = \mathcal{P}_0\,\hat{\mathcal{V}}_0$. The condition $\mathcal{A}_0\hat{\mathcal{V}}_0 = 0$ on $\hat{\mathcal{V}}_0 = (\hat{q}_0, \hat{v}_0, \mathcal{F}\tilde{A}^*_0)$ may be declined in

$$\begin{cases} \xi_\perp \cdot \hat{v}_{\perp 0} = 0, \\ i\,{}^t({}^t\xi_\perp, 0)\,\hat{q}_0 + \psi_0(\xi_\perp)\,J\,P_0(\xi_\perp)\,\mathcal{F}\tilde{A}^*_0 = 0\,, \\ \psi_0(\xi_\perp)\,P_0(\xi_\perp)\,J\,\hat{v}_0 = 0\,. \end{cases}$$

Since

$$P_0(\xi_\perp)\,J\,\hat{v}_0 = -{}^t\big(|\xi_\perp|^{-2}\,(\xi_\perp \cdot \hat{v}_{\perp 0})\,{}^t\xi_\perp^\perp, 0\big)\,,$$

the first and third equation are equivalent. Since

$$J\,P_0(\xi_\perp)\,\mathcal{F}\tilde{A}^*_0 = \ |\xi_\perp|^{-2}\,{}^t\big((\xi_\perp^\perp \cdot \mathcal{F}\tilde{A}^*_{\perp 0})\,{}^t\xi_\perp, 0\big)\,,$$

the second relation is the same as adjusting $\hat{q}_0$ in terms of $\mathcal{F}\tilde{A}^*_{\perp 0}$ according to

$$\hat{q}_0 = i\ \langle \xi_\perp \rangle^{-1}\,\frac{\xi_\perp^\perp}{|\xi_\perp|} \cdot \mathcal{F}\tilde{A}^*_{\perp 0}\,. \tag{4.34}$$

It is apparent that the kernel of $\mathcal{A}_0$ is of dimension 5, given by

$$\mathrm{Ker}\,\mathcal{A}_0 = \big\{ \mathcal{V}_0 \in \mathbb{C}^7\,;\ \xi_\perp \cdot \hat{v}_{\perp 0} = 0 \text{ and } (\hat{q}_0, \mathcal{F}\tilde{A}^*_{\perp 0}) \text{ satisfies } (4.34) \big\}\,.$$

In factor of ε^0, we find

$$\partial_t \hat{\mathcal{V}}_0 + \mathcal{A}_1 \hat{\mathcal{V}}_0 + \mathcal{A}_0 \hat{\mathcal{V}}_1 = 0 .$$

Since $\mathcal{A}_0$ is skew-adjoint $\mathcal{P}_0 \mathcal{A}_0 = 0$, and then applying $\mathcal{P}_0$ to the previous equation, we obtain the ODE

$$\partial_t \mathcal{P}_0 \hat{\mathcal{V}}_0 + (\mathcal{P}_0 \mathcal{A}_1 \mathcal{P}_0) \hat{\mathcal{V}}_0 = 0 . \tag{4.35}$$

We must complete this with what comes from (4.30), that is

$$\mathcal{F}\tilde{A}_0^*(0,\cdot) = P_0(\xi_\perp)\, \mathcal{F}\tilde{A}_0^*(0,\cdot) , \tag{4.36}$$

which is the same as

$$\mathcal{F}\tilde{A}_0^*(0,\cdot) = {}^t(\alpha\, {}^t\xi_\perp^\perp, \beta) , \qquad (\alpha, \beta) \in \mathbb{C} .$$

Since the matrix $\mathcal{P}_0 \mathcal{A}_1 \mathcal{P}_0$ is skew-adjoint, the norm of $\mathcal{P}_0 \hat{\mathcal{V}}_0$ is conserved

$$\| \mathcal{P}_0 \hat{\mathcal{V}}_0(t,\xi) \| \leq \| \mathcal{P}_0 \hat{\mathcal{V}}_0(0,\xi) \| , \qquad \forall t \in \mathbb{R} . \tag{4.37}$$

We can also extract

$$\mathcal{Q}_0 \hat{\mathcal{V}}_1 = -(\mathcal{Q}_0 \mathcal{A}_0 \mathcal{Q}_0)^{-1} (\mathcal{Q}_0 \mathcal{A}_1 \mathcal{P}_0) \hat{\mathcal{V}}_0 .$$

We put in place a proof by induction based on

(H_j) "For $0 \leq k \leq j$, we know $\hat{\mathcal{V}}_k$ and the component $\mathcal{Q}_0 \hat{\mathcal{V}}_{k+1}$"

We have just seen that (H_0) is verified. Now, we fix $j \geq 1$, and we suppose that (H_{j-1}) is satisfied. In factor of ε^j, we obtain

$$\partial_t \hat{\mathcal{V}}_j + \mathcal{A}_1 \mathcal{P}_0 \hat{\mathcal{V}}_j + \mathcal{A}_0 \mathcal{Q}_0 \hat{\mathcal{V}}_{j+1} + \mathcal{A}_1 \mathcal{Q}_0 \hat{\mathcal{V}}_j + \sum_{k=2}^{j+1} \mathcal{A}_k \hat{\mathcal{V}}_{j+1-k} = 0 . \tag{4.38}$$

We project (4.38) through $\mathcal{P}_0$ to eliminate $\hat{\mathcal{V}}_{j+1}$. This yields

$$\partial_t \mathcal{P}_0 \hat{\mathcal{V}}_j + (\mathcal{P}_0 \mathcal{A}_1 \mathcal{P}_0) \hat{\mathcal{V}}_j + \cdots = 0 , \tag{4.39}$$

where the ellipsis "$\cdots$" contains many terms but which, according to (H_{j-1}), have all been identified. Furthermore, to obtain (4.30) up to the order j, it is necessary to adjust $\mathcal{F}\tilde{A}_j^*(0,\cdot)$ in such a way that

$$(\mathrm{Id} - P_0)\, \mathcal{F}\tilde{A}_j^*(0,\cdot) = -\sum_{k=1}^{j} P_k\, \mathcal{F}\tilde{A}_{j-k}^* , \tag{4.40}$$

where the right hand side is a known expression. By solving the (well-posed) Cauchy problem associated with (4.39), we have access to the component $\mathcal{P}_0\,\hat{\mathcal{V}}_j$. Then, we can obtain $\mathcal{Q}_0\,\hat{\mathcal{V}}_{j+1}$ through

$$\mathcal{Q}_0\,\hat{\mathcal{V}}_{j+1} = -(\mathcal{Q}_0\,\mathcal{A}_0\,\mathcal{Q}_0)^{-1}\left[\partial_t(\mathcal{Q}_0\,\hat{\mathcal{V}}_j) + (\mathcal{Q}_0\,\mathcal{A}_1\,\mathcal{P}_0)\,\hat{\mathcal{V}}_j\right] + \cdots .$$

This means that (H_j) is verified. By induction, we can determine all the $\hat{\mathcal{V}}_j$ (up to $j = N$). There remains to check that the L^2-estimates recorded in (4.29) and (4.30) do apply.

As before, to compensate the growth with respect to ξ induced by the $\mathcal{A}_j$, we must start with initial data that are all rapidly decreasing in ξ, at the rate $|\xi|^{-k}$ for all $k \in \mathbb{N}$. In other words, we impose

$$\forall k \in \mathbb{N}, \qquad \exists\, \bar{C}_k^j \in \mathbb{R}_+ ; \qquad |\hat{\mathcal{V}}_j(0,\xi)| \leq \bar{C}_k^j\,(1+|\xi|^2)^{-k/2} . \tag{4.41}$$

This means that the initial data $\mathcal{V}_j(0,\cdot)$ belong to H^k for all k. In addition to this (rather classical) condition, there is a further difficulty.

Indeed, the matrices $\mathcal{A}_j$ imply singularities which are localized along D_v. The definition (4.31) of $\mathcal{A}_1$ indicates that $\mathcal{A}_1$ depends on P_1. On the other hand, in view of (4.26), we have $|P_1| \lesssim |\xi_3|/|\xi_\perp|$. In particular, the size of P_1 may explode when $|\xi_\perp|$ goes to 0. This behavior may be transmitted to $\hat{\mathcal{V}}_1$ (and step by step to all $\hat{\mathcal{V}}_j$) since the equation (4.38) for $j = 1$ involves the product $\mathcal{A}_1\,\mathcal{Q}_0\,\hat{\mathcal{V}}_1$, and thereby $\mathcal{A}_1\,(\mathcal{Q}_0\,\mathcal{A}_0\,\mathcal{Q}_0)^{-1}\,(\mathcal{Q}_0\,\mathcal{A}_1\,\mathcal{P}_0)\,\hat{\mathcal{V}}_0$. To avoid this, we can adjust the initial data in such a way that the $\mathcal{V}_j(0,\cdot)$ are flat along D_v (or even better are zero in a neighborhood of D_v). In other words, we strengthen the condition (4.41) according to

$$\forall (\tilde{k},k) \in \mathbb{N}^2, \qquad \exists\, C_{\tilde{k},k}^j \in \mathbb{R}_+ ; \qquad |\hat{\mathcal{V}}_j(0,\xi_\perp,\xi_3)| \leq C_{\tilde{k},k}^j\,|\xi_\perp|^{\tilde{k}}\,(1+|\xi|^2)^{-k/2} . \tag{4.42}$$

The idea is to cancel the impact of singularities issued from D_v.

Lemma 9. *Let us fix some integer $N \in \mathbb{N}$ and some time $T \in \mathbb{R}_+^*$. For $0 \leq j \leq N$, we select initial data $\hat{\mathcal{V}}_j(0,\cdot)$ polarized as indicated in* (4.36) *and* (4.40)*. We assume moreover that they satisfy* (4.42)*. Then, there exists an approximate solution $\hat{\mathcal{V}}^a$ to* (4.18) *which is of order N, which has the form* (4.28)*, and which matches with the initial conditions $\hat{\mathcal{V}}_j(0,\cdot)$.*

Proof. The construction of the $\hat{\mathcal{V}}_j$ has already been explained. Note that

$$(\mathrm{Id} - P_\varepsilon)\,\mathcal{F}\tilde{A}^{*a}(0,\cdot) = \varepsilon^N \sum_{j=N+1}^{2N} \varepsilon^{j-N} \sum_{k=1}^{j} P_k\,\mathcal{F}\tilde{A}^*_{j-k} .$$

The multiplication by the matrices P_k can result in a loss of the type $|\xi_\perp|^{-\iota(k)}$ for a finite index $\iota(k)$. But this is compensated by (4.42). By this way, we can recover (4.30).

Below, we will not explain (because it is fairly standard) what happens at large frequencies ($|\xi| \gg 1$) from the viewpoint of L^2-estimates. Instead, we dwell on what occurs near D_v. In view of (4.37), the bound (4.42) passes to $\mathcal{V}_0$ according to

$$\forall (\tilde{k},k) \in \mathbb{N}^2, \qquad \forall t \in [0,T], \qquad |\hat{\mathcal{V}}_0(t,\xi_\perp,\xi_3)| \leq C_{\tilde{k},k}^0\,|\xi_\perp|^{\tilde{k}}\,(1+|\xi|^2)^{-k/2} .$$

The multiplication (two times) of $\hat{\mathcal{V}}_0$ by $\mathcal{A}_1$ induces a loss of (at most) $\xi_3^2/|\xi_\perp|^2$. This reveals in the source term of the equation on $\hat{\mathcal{V}}_1$ a bound by

$$\leq C^0_{\tilde{k},k}\, |\xi_\perp|^{\tilde{k}-2}\, \big(\xi_3^2/(1+|\xi|^2)\big)\, (1+|\xi|^2)^{-(k-2)/2} \leq C^0_{\tilde{k},k}\, |\xi_\perp|^{\tilde{k}-2}\, (1+|\xi|^2)^{-(k-2)/2}.$$

Above, we still have (4.42) for $j=0$ just by replacing $C^0_{\tilde{k},k}$ by $C^0_{\tilde{k}+2,k+2}$. It follows that for all $\in \mathbb{N}$, we have

$$\forall t \in [0,T], \quad \exists\, \tilde{C}^1_{\tilde{k},k} \in L^\infty([0,T])\,; \quad |\hat{\mathcal{V}}_1(t,\xi_\perp,\xi_3)| \leq \tilde{C}^1_{\tilde{k},k}\, |\xi_\perp|^{\tilde{k}}\, (1+|\xi|^2)^{-k/2}\,.$$

This argument can be iterated because the multiplication a finite number of times by the weight $\xi_3^2/|\xi_\perp|^2$ does not affect the rates of decrease that have been selected. This leads (for all $j \leq N$ and for all $k \in \mathbb{N}$) to

$$\forall t \in [0,T], \quad \exists\, \tilde{C}^j_{\tilde{k},k} \in L^\infty([0,T])\,; \quad |\hat{\mathcal{V}}_j(t,\xi_\perp,\xi_3)| \leq \tilde{C}^j_{\tilde{k},k}\, |\xi_\perp|^{\tilde{k}}\, (1+|\xi|^2)^{-k/2}\,.$$

Since the remainder $\hat{R}^a_\varepsilon$ is built as a sum of $\mathcal{A}_k$ multiplied by the $\mathcal{V}_k$, it is indeed controlled in L^2 as expected in (4.29). □

4.3.2. Justification of the formal calculus

To obtain exploitable estimates, the only way is to work with a suitable formulation, that is with (4.18).

Lemma 10 *(Stability). Let $\hat{\mathcal{V}}^a_\varepsilon(t,\xi)$ be a given approximate solution to (4.18) which is of order N and which is defined on the interval $[0,T]$. Then, the solution $\hat{\mathcal{V}}_\varepsilon = (\hat{q}_\varepsilon, \hat{v}_\varepsilon, \mathcal{F}\tilde{A}^*_\varepsilon)$ to (4.18) associated with the initial data $\hat{\mathcal{V}}_\varepsilon(0,\cdot) = {}^t(\hat{q}^a_\varepsilon, \hat{v}^a_\varepsilon, \mathcal{F}\tilde{A}^{*a}_\varepsilon)(0,\cdot)$ is, for the constant C appearing in (4.29) and (4.30), controlled by*

$$\| \hat{\mathcal{V}}_\varepsilon - \hat{\mathcal{V}}^a_\varepsilon \|_{L^\infty([0,T];L^2(\mathbb{R}^3))} \leq 3\, C\, (1+T)\, \varepsilon^N. \tag{4.43}$$

Proof. We introduce the auxiliary expression $\check{\mathcal{V}}^a_\varepsilon := {}^t(\hat{q}^a_\varepsilon, \hat{v}^a_\varepsilon, P_\varepsilon \mathcal{F}\tilde{A}^{*a}_\varepsilon)$. By exploiting (4.29) and the structure of $\mathcal{A}_\varepsilon$, we get

$$\partial_t (\mathrm{Id} - P_\varepsilon)\mathcal{F}\tilde{A}^a_\varepsilon = (\mathrm{Id} - P_\varepsilon)\hat{R}^a_{\tilde{A}^*_\varepsilon}\,.$$

By combining (4.29) and (4.30), this furnishes

$$\| \hat{\mathcal{V}}^a_\varepsilon - \check{\mathcal{V}}^a_\varepsilon \|_{L^\infty([0,T];L^2(\mathbb{R}^3))} = \| (\mathrm{Id} - P_\varepsilon)\, \mathcal{F}\tilde{A}^{*a}_\varepsilon \|_{L^\infty([0,T];L^2(\mathbb{R}^3))} \leq C\, (1+T)\, \varepsilon^N. \tag{4.44}$$

We consider now the difference $\mathcal{D}_\varepsilon := \check{\mathcal{V}}^a_\varepsilon - \hat{\mathcal{V}}_\varepsilon$. This time, (4.29) and the properties of $\mathcal{A}_\varepsilon$ lead to

$$\partial_t \check{\mathcal{V}}^a_\varepsilon + \frac{1}{\varepsilon}\, \mathcal{A}_\varepsilon\, \check{\mathcal{V}}^a_\varepsilon = \check{R}^a_\varepsilon := (\hat{R}^a_{q_\varepsilon}, \hat{R}^a_{v_\varepsilon}, P_\varepsilon\, \hat{R}^a_{\tilde{A}^*_\varepsilon})\,.$$

It follows that

$$\partial_t (e^{\mathcal{A}_\varepsilon t/\varepsilon} \, \mathcal{D}_\varepsilon) = e^{\mathcal{A}_\varepsilon t/\varepsilon} \, \check{R}^a_\varepsilon(t) \, , \qquad \mathcal{D}_\varepsilon(0,\cdot) = {}^t\big(0, 0, (P_\varepsilon - \mathrm{Id}) \mathcal{F} \tilde{A}^{*a}_\varepsilon\big).$$

After integration, there remains

$$\mathcal{D}_\varepsilon(t,\xi) = e^{-\mathcal{A}_\varepsilon t/\varepsilon} \, \mathcal{D}_\varepsilon(0,\xi) + \int_0^t e^{\mathcal{A}_\varepsilon (r-t)/\varepsilon} \, \check{R}^a_\varepsilon(r,\xi) \, dr \, .$$

Since $\mathcal{A}_\varepsilon$ is diagonalizable with purely imaginary eigenvalues, we have

$$|\mathcal{D}_\varepsilon(t,\xi)| \leq |\mathcal{D}_\varepsilon(0,\xi)| + \int_0^t |\check{R}^a_\varepsilon(r,\xi)| \, dr \, .$$

We integrate with respect to ξ to obtain (using the Cauchy-Schwarz inequality)

$$\begin{aligned} \int_{\mathbb{R}^3} |\mathcal{D}_\varepsilon(t,\xi)|^2 \, d\xi \; &\leq 2 \int_{\mathbb{R}^3} |\mathcal{D}_\varepsilon(0,\xi)|^2 \, d\xi + 2 \int_{\mathbb{R}^3} t \int_0^t |R^a_\varepsilon(r,\xi)|^2 \, dr \, d\xi \\ &\leq 2 \, C^2 \, \varepsilon^{2N} \, (1+t^2) \, . \end{aligned}$$

In particular

$$\| \, \mathcal{D}_\varepsilon \, \|_{L^\infty([0,T];L^2(\mathbb{R}^3))} \leq 2 \, C \, (1+T) \, \varepsilon^N \, . \tag{4.45}$$

By summing (4.44) and (4.45), we recover (4.43). □

Recall that $\mathcal{V}_\varepsilon = (q_\varepsilon, v_\varepsilon, \tilde{A}^*_\varepsilon) = \mathcal{F}^{-1} \hat{\mathcal{V}}_\varepsilon$ is a solution to the system (4.21). Since the Fourier transformation conserves (modulo a constant) the L^2-norm, the expression

$$\mathcal{V}^a_\varepsilon := \mathcal{F}^{-1} \hat{\mathcal{V}}^a_\varepsilon = \mathcal{V}_0 + \varepsilon \, \mathcal{V}_1 + \cdots + \varepsilon^N \, \mathcal{V}_N \, , \qquad \mathcal{V}_j = (q_j, v_j, \tilde{A}^*_j) := \mathcal{F}^{-1} \hat{\mathcal{V}}_j \, ,$$

furnishes an approximate solution of (4.21) in the sense that

$$\| \, \mathcal{V}_\varepsilon - \mathcal{V}^a_\varepsilon \, \|_{L^\infty([0,T];L^2(\mathbb{R}^3))} \lesssim (1+T) \, \varepsilon^N \, .$$

This provides an algorithm to construct (a kind of) approximate solutions to (4.21), as well as arguments to justify them.

4.3.3. Return to the potential formulation

The preceding computations can be interpreted at the level of (4.1). To this end, it suffices to pass from $\tilde{A}^*$ to A^* through $A^* = \psi_\varepsilon(D)^{-1}\tilde{A}^*$. As long as $\xi_\perp \neq 0$, we can use a Taylor expansion (up to any order N) of $\psi(\varepsilon,\cdot)^{-1}$, that is

$$\psi(\varepsilon,\xi)^{-1} = \chi_0(\xi) + \sum_{n=2}^{N} \varepsilon^n \, \chi_n(\xi) + O(\varepsilon^{N+1}) \,, \qquad \chi_0 := \psi_0^{-1} \,. \tag{4.46}$$

From there, we can reconstitute

$$A^* = \psi_\varepsilon(D)^{-1}\tilde{A}^* = \sum_{j=0}^{N} \varepsilon^j \, A_j^* + O(\varepsilon^{N+1}) \,, \qquad A_j^* := \sum_{k=0}^{j} \chi_k(D)\, \tilde{A}_{j-k}^* \,.$$

However, it must be remembered that the χ_k explode (at an algebraic rate) near $\xi_\perp = 0$. As long as the $\tilde{A}_k^*$ are flat along D_v, the products $\chi_k(D)\,\tilde{A}_{j-k}^*$ make sense. This works due to (4.42). Otherwise, that is when hypotheses (4.42) are not assumed, this may fail. Indeed, in Subsection 4.3, we have investigated (4.1), with A^* in the form $A^* = A_0^* + \varepsilon\, A_1^* + \cdots$, which furnishes $\tilde{A}^* = \tilde{A}_0^* + \varepsilon\,\tilde{A}_1^* + \cdots$ with $\tilde{A}_0^* = \psi_0(D)\, P_0(D)\, A_0^*$. The multiplication on the Fourier side by $\psi_0(\xi_\perp)$ implies that $\mathcal{F}\tilde{A}_0^*$ is sure to become flat along D_v.

But such a cancellation is not evident concerning the $\tilde{A}_j^*$ with $j \geq 1$, when activating a WKB calculus up to any order. Still, this explains why, at the order 0, it is still possible to extract at the level of (4.1) a modulation equation for the main profile (without pre-empting similar results for $N \geq 1$).

As previously discussed, the condition (4.42) ensures the existence of *certain* approximate solutions to (4.1) with any desired precision $N \in \mathbb{N}$. However, this approach is limited in the context of (1.1) due to the presence of nonlinearities. Indeed, the interaction of functions localized in frequency away from $\xi_\perp = 0$ can produce non-zero terms along D_v, thereby causing singularities to emerge. Consequently, in the case of (1.1), the analysis is limited to the order $N = 1$.

5. Construction of exact solutions

The aim of this section is to prove Theorem 1. To achieve this, we must implement uniform (with respect to $\varepsilon \in]0,1]$) energy estimates for a version of the full system (1.1)-(1.2)-(1.3). This uniformity is crucial for obtaining a uniform lifespan $T \in \mathbb{R}_+^*$ and is also an essential step in showing (1.10) and (1.11). The preceding Sections 3 and 4 provide a useful foundation for this endeavor. The method outlined in Subsection 3.2 for the modulation equations offers a clear guide on how to proceed, while Subsections 4.2 and 4.3 highlight potential pitfalls (related to singularities) that must be avoided. To establish control over U_ε, we employ the change of variables (4.17) once again. The resulting system is presented in Subsection 5.1, and the corresponding energy estimates are detailed in Subsection 5.2. Finally, the proof of Theorem 1 is completed in Subsection 5.3.

5.1. Transformation of the system

We follow the general guidelines outlined in Paragraph 3.2.3, with some necessary modifications. The first step, informed by the insights gained in Subsection 4.1, is to replace (1.1) with a quasilinear version whose singular part is skew-adjoint. Taking into account the role of ψ_ε in the analysis of the toy model (4.1), we substitute A^* for $\tilde{A}^*$ as indicated in (4.15). Of course, the relation (1.2) gives rise to

$$\frac{1}{\varepsilon}\, \nabla_\varepsilon \cdot \tilde{A}^* = 0\,. \tag{5.1}$$

It is more complicated to interpret the constitutive relation (1.3) in terms of $\tilde{A}^*$. This is addressed in the next paragraph.

5.1.1. Replacement of the difference $A^* - A$ by an operator acting on $\tilde{A}^*$

The first difficulty is about the interpretation of $A^* - A$ as depending on $\tilde{A}^*$. It is a matter of replacing (3.17). The constitutive relation (1.3) is the same as $A^* = A + \rho^{-1}\, \nabla_\varepsilon \times (\nabla_\varepsilon \times A)$ so that

$$\begin{aligned} A^* - A \;&= \big(\mathrm{Id} - (\mathrm{Id} + \rho^{-1}\, \nabla_\varepsilon \times \nabla_\varepsilon \times)^{-1}\big) A^* \\ &= (\mathrm{Id} + \rho^{-1}\, \nabla_\varepsilon \times \nabla_\varepsilon \times)^{-1}\, \rho^{-1}\, \nabla_\varepsilon \times (\nabla_\varepsilon \times A^*) \\ &= (\rho\, \mathrm{Id} + \nabla_\varepsilon \times \nabla_\varepsilon \times)^{-1}\, \nabla_\varepsilon \times (\nabla_\varepsilon \times A^*)\,. \end{aligned}$$

Recall that $\bar{\rho} = 1$. Consider the differential operator

$$\mathscr{L} \equiv \mathscr{L}(\rho, D) := \rho\, \mathrm{Id} + \nabla_\varepsilon \times \nabla_\varepsilon \times \qquad \overline{\mathscr{L}} \equiv \mathscr{L}(\bar{\rho}, D) := \mathrm{Id} + \nabla_\varepsilon \times \nabla_\varepsilon \times$$

From [4], we know that the matrix valued operator $\mathscr{L}$ is self-adjoint and L^2-coercive (as long as the density ρ stays away from 0). It is therefore invertible. Its inverse which is denoted by $\mathscr{L}^{-1}$ is bounded in L^2. Since the vector field A^* is divergence free and because $\psi_\varepsilon(D)^2 = -(1-\Delta_\varepsilon)^{-1}\, \Delta_\varepsilon$, we have

$$\nabla_\varepsilon \times \nabla_\varepsilon \times A^* = -\Delta_\varepsilon A^* = (1-\Delta_\varepsilon)\, \psi_\varepsilon(D)\, \tilde{A}^*.$$

This means that

$$A^* - A = \mathrm{K}\, \tilde{A}^*, \qquad \mathrm{K} \equiv \mathrm{K}(\rho, D) := \mathscr{L}^{-1}\, (1-\Delta_\varepsilon)\, \psi_\varepsilon(D)\,.$$

In view of Proposition 13 in [4], the action of $\mathscr{L}^{-1}$ is given by a pseudo-differential operator which is of order -2 on solenoidal fields. Thus, the restriction of K on divergence free vector fields (such as $\tilde{A}^*$) is associated with a pseudo-differential operator of order 0. It is therefore continuous on L^2.

Lemma 11 *(Refined description of* K *when applied to solenoidal vector fields). Assume that* $\nabla_\varepsilon \cdot \tilde{A}^* = 0$ *and that* $q \in H^s(\Omega)$ *with* $s > 5/2$ *is such that*

$$\exists\, c \in \mathbb{R}_+^*\,; \qquad 0 < c \le \bar{\mathrm{q}} + \varepsilon\, q(x)\,, \quad \forall\, (\varepsilon, x) \in [0,1] \times \Omega\,.$$

Then, the action of K *on such* $\tilde{A}^*$ *can be decomposed into*

$$\mathrm{K} = \psi_\varepsilon(D) + \varepsilon\, \mathcal{K}\,, \qquad \mathcal{K} \equiv \mathcal{K}_\varepsilon(q, D) := \mathscr{L}^{-1}\, \chi(\varepsilon, q)\, \psi_\varepsilon(D)\,, \tag{5.2}$$

where χ *is a smooth function with respect to* $(\varepsilon, q) \in [0, 1] \times \mathbb{R}$*, and where* $\mathcal{K}$ *is uniformly bounded on* L^2*. In other words*

$$\exists C \in \mathbb{R}_+^*\,; \qquad \| \mathcal{K}\, \tilde{A}^* \|_{L^2} \leq C\, \| \tilde{A}^* \|_{L^2}\,, \quad \forall \varepsilon \in [0, 1]\,. \tag{5.3}$$

Proof. Define $\bar{\mathrm{K}} := \mathrm{K}(\bar{\rho}, D)$. Since $\nabla_\varepsilon \cdot \tilde{A}^* = 0$, we have $\bar{\mathrm{K}}\, \tilde{A}^* = \psi_\varepsilon(D)\, \tilde{A}^*$. By construction (with $\bar{\rho} = 1$):

$$A^* - A = \bar{\mathrm{K}}\, \tilde{A}^* + (\mathrm{K} - \bar{\mathrm{K}})\, \tilde{A}^* = \psi_\varepsilon(D)\, \tilde{A}^* + (\mathscr{L}^{-1} - \overline{\mathscr{L}}^{-1})\, (1 - \Delta_\varepsilon)\, \psi_\varepsilon(D)\, \tilde{A}^*.$$

Observe that

$$\mathscr{L} = \overline{\mathscr{L}} + \varepsilon\, \chi(\varepsilon, q)\,, \qquad \chi(\varepsilon, q) := \int_0^q (\mathrm{g}^{-1})'(\bar{\mathrm{q}} + \varepsilon\, s)\, ds\,.$$

It follows that

$$\mathscr{L}^{-1} - \overline{\mathscr{L}}^{-1} = \mathscr{L}^{-1}\, (\overline{\mathscr{L}} - \mathscr{L})\, \overline{\mathscr{L}}^{-1} = -\,\varepsilon\, \mathscr{L}^{-1}\, \chi(\varepsilon, q)\, \overline{\mathscr{L}}^{-1}. \tag{5.4}$$

Recall that $\overline{\mathscr{L}} = 1 - \Delta_\varepsilon$ on divergence free vector fields. Then, it suffices to exploit (5.4) to make appear (5.2). Since $s > 5/2$, the multiplication operator by $\chi(\varepsilon, q) \in H^s$ is bounded in L^2. Hence, we have (5.3). □

The operator K is not self-adjoint. To remedy this, we put aside $\psi_\varepsilon(D)$, and we write K in the form

$$\mathrm{K} = (1 - \Delta_\varepsilon)^{-1}\, \mathrm{M}\, \psi_\varepsilon(D)\,, \qquad \mathrm{M} \equiv \mathrm{M}_\varepsilon(q, D) := (1 - \Delta_\varepsilon)\, \mathscr{L}^{-1}\, (1 - \Delta_\varepsilon)\,.$$

Introduce $\bar{\mathrm{M}} := \mathrm{M}(\bar{q}, D) = 1 - \Delta_\varepsilon = \bar{\mathrm{M}}^*$. It is clear that M, and therefore $\mathcal{M} := (\mathrm{M} - \bar{\mathrm{M}})/\varepsilon$, is self-adjoint. Compute

$$\mathcal{M} := (1 - \Delta_\varepsilon)\, \mathscr{L}^{-1}\, \chi(\varepsilon, D) = \mathcal{M}^* = \chi(\varepsilon, D)\, \mathscr{L}^{-1}\, (1 - \Delta_\varepsilon)\,.$$

Instead of looking at $\mathcal{M}$, we consider $\mathcal{M}^*$. The advantage is that χ appears now on the left of $\mathscr{L}^{-1}$. As a consequence, we can assert that $\mathscr{L}^{-1}\, (1 - \Delta_\varepsilon)$ acts on solenoidal vector fields as a uniformly bounded L^2-operator. As soon as $q \in H^s$ with $s > 5/2$, the same applies concerning $\chi(\varepsilon, D)\, \mathscr{L}^{-1}\, (1 - \Delta_\varepsilon)$. This implies that the actions of $\mathcal{M} : L^2 \to L^2$ and hence of $\mathcal{K} = (1 - \Delta_\varepsilon)^{-1}\, \mathcal{M}\, \psi_\varepsilon(D) : L^2 \to H^2$ are uniformly bounded. The information (5.3) is not optimal but it is sufficient for present purposes.

5.1.2. A change of state variables

We multiply the last line of (1.1) by $\psi_\varepsilon(D)\,P_\varepsilon(D)$ in order to obtain a self-contained system on $\tilde U^r \equiv \tilde U^r_\varepsilon := (q_\varepsilon, v_\varepsilon, \tilde A^*_\varepsilon)$. Taking into account the relation (5.2), we find

$$\begin{cases} \partial_t q + (v\cdot\nabla_\varepsilon)q + \dfrac{1}{\varepsilon}\,\mathrm{a}(\bar{\mathrm{q}}+\varepsilon\, q)\;\nabla_\varepsilon\cdot v = 0\,, \\ \partial_t v + (v\cdot\nabla_\varepsilon)v + \dfrac{1}{\varepsilon}\,\mathrm{a}(\bar{\mathrm{q}}+\varepsilon\, q)\;\nabla_\varepsilon q - \dfrac{1}{\varepsilon}\;\psi_\varepsilon(D)\tilde A^*\times e_z - \mathcal{K}\tilde A^*\times e_z \\ \qquad -\mathrm{K}\,\tilde A^*\times\big(\nabla_\varepsilon\times\psi_\varepsilon(D)^{-1}\tilde A^*\big) + \nabla_\varepsilon\Big(\dfrac{|\mathrm{K}\,\tilde A^*|^2}{2}\Big) = \nu\;\nabla_\varepsilon(\nabla_\varepsilon\cdot v)\,, \\ \partial_t\tilde A^* - \psi_\varepsilon(D)\,P_\varepsilon(D)\big((v-\mathrm{K}\,\tilde A^*)\times(\nabla_\varepsilon\times\psi_\varepsilon(D)^{-1}\tilde A^*)\big) \\ \qquad -\psi_\varepsilon(D)\,P_\varepsilon(D)\,\big(\mathrm{K}\,\tilde A^*\times(\nabla_\varepsilon\times v)\big) - \dfrac{1}{\varepsilon}\;\psi_\varepsilon(D)\,P_\varepsilon(D)\,(v\times e_z) \\ \qquad +\dfrac{1}{\varepsilon}\psi_\varepsilon(D)P_\varepsilon(D)(\psi_\varepsilon(D)\tilde A^*\times e_z) + \psi_\varepsilon(D)\,P_\varepsilon(D)\,(\mathcal{K}\,\tilde A^*\times e_z) = 0\,. \end{cases} \tag{5.5}$$

The set of equations on $\tilde U^r_\varepsilon$ built with (5.1)-(5.5) is self-contained. It can be written in the abbreviated form $\tilde{\mathcal{L}}(\tilde U^r_\varepsilon;\partial)\tilde U^r_\varepsilon = 0$. With J as in (4.19), we can identify the penalized part:

$$\tilde{\mathcal{A}}_\varepsilon(D) := \begin{pmatrix} 0 & \nabla_\varepsilon\cdot & 0 \\ \nabla_\varepsilon & 0 & \psi_\varepsilon(D)\,J\,P_\varepsilon(D) \\ 0 & P_\varepsilon(D)\,J\,\psi_\varepsilon(D) & (P_\varepsilon\,\psi_\varepsilon)(D)\,J\,(\psi_\varepsilon\,P_\varepsilon)(D) \end{pmatrix} = -\tilde{\mathcal{A}}^\dagger_\varepsilon(D)\,. \tag{5.6}$$

The block at the bottom right of $\tilde{\mathcal{A}}_\varepsilon(D)$ comes in fact from the Hall effect. It occurs in Section 3 through the influence of $(A^*_{1\perp} - A_{1\perp})^\perp$ inside (3.8). But it was not incorporated in the toy model of Section 4. To see why, just compare $\tilde{\mathcal{A}}_\varepsilon(D)$ with the matrix $\tilde{\mathcal{A}}_\varepsilon(\xi)$ which was introduced inside (4.19). Now, observe that this block is skew-adjoint. For this reason, it raises no particular problems. As explained in Subsection 4.1, the substitution of A by $\tilde A$ is still the gateway to uniform L^2-estimates.

However, the foregoing symmetrization procedure induces a change (when compared to [4]) of the other terms, those of size $O(1)$. The structure of these extra contributions is quite uncommon. It shall be ensured that L^2-estimates remain available, including when dealing with these new differential effects. This issue is examined in the following subsection.

5.2. Energy estimates on the reduced system

We consider the closed system made of (5.1)-(5.5). We assume that the coefficients in factor of derivatives, those which are issued from the non linearities, are in H^s. This implies that all components of $\tilde U^r_\varepsilon$ are Lipschitz. From there, we want to control the L^2-norm of $\tilde U^r_\varepsilon(t,\cdot)$ from the one of $\tilde U^r_\varepsilon(0,\cdot)$. To this end, we multiply (5.5) on the left by ${}^t\tilde U^r_\varepsilon$ in order to perform L^2-energy estimates. When doing this, we can rule out the symmetric quasilinear terms of the two first lines and the impact of $\tilde{\mathcal{A}}_\varepsilon(D)$ which both can be dealt with classical arguments. On the other hand, with (5.3), we can infer that

$$\left|\int_\Omega v\cdot(\mathcal{K}\tilde{A}^*\times e_z)\,dx\right|+\left|\int_\Omega \tilde{A}^*\cdot\big((\psi_\varepsilon(D)\,\mathcal{K}\tilde{A}^*\times e_z\big)\,dx\right|\lesssim\big(\|\,v\,\|_{L^2}+\|\,\tilde{A}^*\,\|_{L^2}\big)\;\|\,\tilde{A}^*\,\|_{L^2}\,.$$

By Young inequality, we get that

$$\begin{aligned}\int_\Omega v\cdot\nabla_\varepsilon|\mathrm{K}\,\tilde{A}^*|^2\,dx\;\; -\frac{\nu}{2}\int_\Omega v\cdot\nabla_\varepsilon(\nabla_\varepsilon\cdot v)\,dx&\\ &=-\int_\Omega(\nabla_\varepsilon\cdot v)\,|\mathrm{K}\,\tilde{A}^*|^2\,dx+\frac{\nu}{2}\int_\Omega(\nabla_\varepsilon\cdot v)^2\,dx\\ &\leq\frac{1}{2\,\nu}\int_\Omega|\mathrm{K}\,\tilde{A}^*|^4\,dx\leq\frac{C}{\nu}\;\|\,\mathrm{K}\,\tilde{A}^*\,\|^2_{L^\infty}\int_\Omega|\mathrm{K}\,\tilde{A}^*|^2\,dx\\ &\lesssim\frac{C}{\nu}\;\|\,\mathrm{K}\,\tilde{A}^*\,\|^2_{H^{s-1}}\;\|\,\tilde{A}^*\,\|^2_{L^2}\,.\end{aligned}$$

Knowing that both ρ and $\tilde{A}^*$ are in a ball of H^s with radius R, we claim that (with a constant depending on R)

$$\|\,\mathrm{K}\,\tilde{A}^*\,\|^2_{H^{s-1}}\lesssim 1. \tag{5.7}$$

To prove this bound, for $|\alpha|\leq s-1$ and $s\in\mathbb{N}^*$ with $s>5/2$, we have to estimate $\partial_x^\alpha(\mathrm{K}\tilde{A}^*)$ in L^2. Consider first the case $|\alpha|=1$. From Lemma 11, we can assert that

$$\|\,\partial_j\,(\mathrm{K}\tilde{A}^*)\,\|_{L^2}\leq\|\,[\partial_j;\mathrm{K}]\,\tilde{A}^*\,\|_{L^2}+\|\,\mathrm{K}\,\partial_j\,\tilde{A}^*\,\|_{L^2}\leq\|\,[\partial_j;\mathrm{K}]\,\tilde{A}^*\,\|_{L^2}+C\;\|\,\tilde{A}^*\,\|_{H^{s-1}}\,.$$

Next, coming back to the definition of $\mathscr{L}$ and K, we find

$$[\partial_j;\mathrm{K}]=[\partial_j;\mathscr{L}^{-1}]\,(1-\Delta_\varepsilon)\,\psi_\varepsilon(D)=\mathscr{L}^{-1}[\mathscr{L};\partial_j]\mathscr{L}^{-1}(1-\Delta_\varepsilon)\,\psi_\varepsilon(D)=\mathscr{L}^{-1}[\rho;\partial_j]\,\mathrm{K}\,,$$

so that (with Lemma 11 again)

$$\|\,[\partial_j;\mathrm{K}]\,\tilde{A}^*\,\|_{L^2}=\|\,\mathscr{L}^{-1}\;\partial_j\rho\,\mathrm{K}\tilde{A}^*\,\|_{L^2}\lesssim\|\,\partial_j\rho\,\|_{L^\infty}\;\|\,\tilde{A}^*\,\|_{L^2}\lesssim\|\,\tilde{U}^r\,\|^2_{H^s}\,.$$

Thus, this works for $|\alpha|=1$. Now, we proceed by iteration. We assume that $\partial_x^\alpha(\mathrm{K}\tilde{A}^*)$ is controlled in L^2 for all $|\alpha|\leq s-2$. For higher order derivatives α with $|\alpha|=s-1$, following the same lines as above, we find that

$$\|\,[\partial_x^\alpha;\mathrm{K}]\,\tilde{A}^*\,\|_{L^2}=\|\,\mathcal{L}^{-1}\;[\rho;\partial_x^\alpha]\,\mathrm{K}\tilde{A}^*\,\|_{L^2}\lesssim\sum_{0<\beta\leq\alpha}\|\,\partial_x^\beta\rho\;\partial_x^{\alpha-\beta}\,\mathrm{K}\tilde{A}^*\,\|_{L^2}\,.$$

On the one hand, since $\rho\in H^s$ and because $|\alpha-\beta|<|\alpha|$ in the above sum, by induction hypotheses, we have

$$\partial_x^\beta\rho\in H^{s-|\beta|}\,,\qquad\partial_x^{\alpha-\beta}\,\mathrm{K}\tilde{A}^*\in H^{|\beta|}.$$

Remark that $(s - |\beta|) + |\beta| = s > 3/2$. Thus, we can exploit the pointwise multiplication rules in Sobolev spaces to control the right hand side as indicated in (5.7). For $s \in \mathbb{R} \setminus \mathbb{N}$, an argument of interpolation is needed.

We now resume in the context of ψ_ε the arguments a) and b) which have been presented in Subsection 3.2.3.

ã) Elimination of the presence of $\psi_\varepsilon(D)$ as acting on the left hand side in the penultimate line of (5.5). Given three scalar functions a, b and c, with $\partial_1 = \partial_x$, $\partial_2 = \partial_y$ and $\partial_3 = \partial_z$, this means to evaluate expressions looking like

$$\int_\Omega a\ \psi_\varepsilon(D)\big[b\ \varepsilon^{\iota_j}\ \partial_j c\big]\,dx\,, \qquad \iota_j := \begin{cases} 0 & \text{if} \quad j \in \{1,2\}\,, \\ 1 & \text{if} \quad j = 3\,. \end{cases}$$

Repeated integrations by parts give rise to

$$\begin{aligned} \int_\Omega a\ \psi_\varepsilon(D)\big[b\ \varepsilon^{\iota_j}\ \partial_j c\big]\ \ dx &= -\int_\Omega \varepsilon^{\iota_j}\ \partial_j\big[b\ \psi_\varepsilon(D)a\big]\ c\ dx = \int_\Omega a\ b\ \varepsilon^{\iota_j}\ \partial_j c\ dx \\ &\quad + \int_\Omega b\ c\ \big(1 - \psi_\varepsilon(D)\big)\ \varepsilon^{\iota_j}\ \partial_j a\ dx + \int_\Omega \varepsilon^{\iota_j}\ \partial_j b\ c\ \big(1 - \psi_\varepsilon(D)\big) a\ dx\,. \end{aligned}$$

The operator norm of $1 - \psi_\varepsilon(D) : L^2 \to L^2$ is clearly less than one. On the other hand, since $\varepsilon^{\iota_j}\,|\xi_j| \le |\xi_\varepsilon|$ for all $j \in \{1,2,3\}$, the symbol of $\big(1 - \psi_\varepsilon(D)\big)\ \varepsilon^{\iota_j}\ \partial_j$ can be bounded as indicated below

$$\left|\left(1 - \frac{|\xi_\varepsilon|}{\langle \xi_\varepsilon \rangle}\right)\ \varepsilon^{\iota_j}\ \xi_j\right| = \frac{1}{\langle \xi_\varepsilon \rangle}\ \frac{\varepsilon^{\iota_j}\ |\xi_j|}{\langle \xi_\varepsilon \rangle + |\xi_\varepsilon|} \le \frac{1}{\langle \xi_\varepsilon \rangle} \le 1\,. \tag{5.8}$$

As a consequence, we have

$$\left|\int_\Omega a\ \psi_\varepsilon(D_\perp)\big[b\ \varepsilon^{\iota_j}\ \ \partial_j c\big]\,dx - \int_\Omega a\ b\ \varepsilon^{\iota_j}\ \ \partial_j c\ dx\right| \lesssim \|\ b\ \|_{W^{1,\infty}}\ \|\ a\ \|_{L^2}\ \|\ c\ \|_{L^2}\,.$$

$\tilde{\text{b}}$) Similarly, the symbol of $\big(1 - \psi_\varepsilon(D)^{-1}\big)\ \varepsilon^{\iota_j}\ \partial_j$ can be estimated according to

$$\left|\left(1 - \frac{\langle \xi_\varepsilon \rangle}{|\xi_\varepsilon|}\right)\ \varepsilon^{\iota_j}\ \xi_j\right| = \frac{\varepsilon^{\iota_j}\ |\xi_j|}{|\xi_\varepsilon|}\ \frac{1}{\langle \xi_\varepsilon \rangle + |\xi_\varepsilon|} \le \frac{1}{\langle \xi_\varepsilon \rangle + |\xi_\varepsilon|} \le 1\,. \tag{5.9}$$

This implies that $\nabla_\varepsilon \times \psi_\varepsilon(D)^{-1}$ can be replaced in the quasilinear terms by $\nabla_\varepsilon \times$ modulo errors which are less than $\|\ \tilde{U}\ \|_{W^{1,\infty}}\ \|\ \tilde{U}\ \|^2_{L^2}$.

After such background work, there remains to perform L^2-energy estimates at the level of the following reduced system

$$\begin{cases} \partial_t v - \mathrm{K}\,\tilde{A}^* \times \big(\nabla_\varepsilon \times \tilde{A}^*\big) = \dfrac{3}{4}\ \nu\ \nabla_\varepsilon(\nabla_\varepsilon \cdot v)\,, \\ \partial_t \tilde{A}^* - (v - \mathrm{K}\,\tilde{A}^*) \times (\nabla_\varepsilon \times \tilde{A}^*) - \mathrm{K}\,\tilde{A}^* \times (\nabla_\varepsilon \times v) = 0\,. \end{cases} \tag{5.10}$$

Let us introduce the Helmholtz–Hodge decomposition of the velocity field v, that is $v = v_l + v_t$ with $\nabla_\varepsilon \cdot v_l = \nabla_\varepsilon \cdot v$ and $\nabla_\varepsilon \times v_t = \nabla_\varepsilon \times v$. The situation concerning the above remaining quasi-linear terms is exactly as in Lemma 19 of [4]. We do not provide details. We only mention the difficulties and the way to circumvent them. Given a vector field C, we consider the differential action $\mathcal{T}_C \cdot := \nabla_\varepsilon \times (C \times \cdot)$. The system (5.10) involves $\mathcal{T}_C^*$ with $C = \mathrm{K}\,\tilde{A}^*$ or $C = v - \mathrm{K}\,\tilde{A}^*$. However, when performing energy estimates, an integration by parts reveals $\mathcal{T}_C$. The operator $\mathcal{T}_C$ is (modulo terms of order 0) skew-adjoint when acting on solenoidal fields (see Remark 12 in [4]). Now, both fields $\tilde{A}^*$ and v_t are divergence free. On the other hand, the L^2-norm of (one order) derivatives of v_l are controlled by the L^2-norm of $\nabla_\varepsilon v$, which is compensated by the bulk viscosity. The condition $\nu \in \mathbb{R}_+^*$ is essential for this to work. Again, we refer to the text [4] for further clarification.

Summary. In brief, we have proved that (for some function F)

$$\frac{d}{dt} \parallel \tilde{U}^r \parallel_{L^2}^2 \lesssim \nu^{-1}\, F\big(\parallel \tilde{U}^r \parallel_{H^s}\big) \, \parallel \tilde{U}^r \parallel_{L^2}^2 \; .$$

By Grönwall's inequality, the L^2-norm of $\tilde{U}^r$ remains under control.

5.3. *Proof of Theorem 1*

The local existence of solutions to (1.1)-(1.2)-(1.3) on $[0, T_\varepsilon[$ for some $T_\varepsilon \in \mathbb{R}_+^*$ depending on $\varepsilon \in]0, 1]$ has been established in [4]. To guarantee (1.9), the strategy is to pass through the formulation (5.1)-(5.5).

As proved in Subsection 5.2, L^2-estimates which are uniform with respect to $\varepsilon \in]0, 1]$ are available. This information is not sufficient. Given some index of regularity s large enough (say $s > 5/2$), uniform H^s-estimates are also needed.

It is a question of commuting (5.5) with spatial derivatives. The system (5.5) is penalized by the linear operator $\tilde{\mathcal{A}}_\varepsilon(D)$ given in (5.6), which is skew-adjoint and with constant coefficients (this is just a Fourier multiplier which does not depend on t and x). As such, the action of $\tilde{\mathcal{A}}_\varepsilon(D)$ does commute with derivatives.

Also notable is the constant identity matrix Id which stands in (5.5) in factor of $\partial_t \tilde{U}_\varepsilon^r$. Applying ∂_j to (5.5), we just find

$$\partial_t(\partial_j \tilde{U}_\varepsilon^r) + \cdots + \frac{1}{\varepsilon}\, \tilde{\mathcal{A}}_\varepsilon(D)\,(\partial_j \tilde{U}_\varepsilon^r) = 0\,,$$

while the condition $\tilde{\mathcal{A}}_\varepsilon(D)\,\tilde{U}_\varepsilon^r = O(\varepsilon)$ implies that $\tilde{\mathcal{A}}_\varepsilon(D)\,(\partial_j \tilde{U}_\varepsilon^r) = O(\varepsilon)$. Thus, there is no need to consider mixed time-space derivatives (which usually require a complicated work of preparation concerning the data). As a consequence, the notion of prepared data can be reduced to Definition 3.

On this basis, the discussion about (5.1)-(5.5) falls within the framework of classical nonlinear optics [25,27], see especially Chapter 4 in the book [25]. Both issues related to existence and stability can easily be answered in this context. Let $s > 5/2$. Select a prepared data which is in H^s. In other words, to solve the Cauchy problem (5.1)-(5.5) on $[0, T]$ with $T > 0$, that is locally in time uniformly with respect to $\varepsilon \in]0, 1]$, it suffices to apply the Theorem 4.4 of [25]. Moreover, the solution $\tilde{U}_\varepsilon^r$ is uniformly stable with respect to H^s-perturbations.

The second part of Theorem 1 deals with the justification (at leading order) of the modulation equation. The formal calculus of Section 3 has been developed on the basis of (1.1)-(1.2)-(1.3). Now, consider the approximate solution $U^a_\varepsilon = U_0 + \varepsilon\, U_1$ presented in Section 3. From there, we can define $\tilde U^{ra}_\varepsilon := \big(q^a_\varepsilon, v^a_\varepsilon, \psi_\varepsilon(D) A^a_\varepsilon\big)$. Since (5.5) is deduced from the approximation $\mathcal{L}(U^a_\varepsilon, \partial) U^a_\varepsilon = O(\varepsilon)$ just by applying the uniformly L^2-bounded operator $(P_\varepsilon\, \psi_\varepsilon)(D)$ to the last equation of (1.1), we still have $\tilde{\mathcal{L}}(\tilde U^{ra}_\varepsilon, \partial)\tilde U^{ra}_\varepsilon = O(\varepsilon)$. The expression $\tilde U^{ra}_\varepsilon$ turns out to be an approximate solution to the system (5.1)-(5.5), while the initial data $\tilde U^{in} := (q^{in}, v^{in}, \tilde A^{in*})$ where $\tilde A^{in*} := \psi_\varepsilon(D)\, A^{in*}$ (and its derivatives) is prepared with respect to $\tilde{\mathcal{L}}$.

Observe that the approximate solution $\tilde U^{ra}_\varepsilon$ to (5.1)-(5.5) has not been obtained by directly looking at (5.5) and by expanding the coefficients of (5.5) in powers of ε. As a matter of fact, the expression $\psi_\varepsilon(\xi)$ cannot be developed in powers of ε without introducing singularities in ξ along the vertical line D_v, which would compromise such a procedure. Instead, we have first constructed an approximate solution to (1.1)-(1.2)-(1.3), and then we have deduced from it the content of $\tilde U^{ra}_\varepsilon$. The justification is achieved at the level of (5.5). But the construction of approximate solutions goes through (1.1) and then is transferred to (5.5). Knowing that (5.1)-(5.5) is stable, we can assert that

$$\| \tilde U^r_\varepsilon - \tilde U^{ra}_\varepsilon \|_{L^\infty([0,T];H^s(\Omega))} = O(\varepsilon)\,. \tag{5.11}$$

Since $\tilde U^r_\varepsilon = {}^t(q_\varepsilon, v_\varepsilon, \tilde A^*_\varepsilon)$ and $\tilde U^{ra}_\varepsilon := \big(q^a_\varepsilon, v^a_\varepsilon, \psi_\varepsilon(D)A^a_\varepsilon\big)$, this already offers immediate access to (1.10). To recover A^*_ε from $\tilde A^*_\varepsilon$, it suffices to apply $\psi_\varepsilon(D)^{-1}$ to $\tilde A_\varepsilon$. However, this may result in a large factor ε^{-1} (see Subsection 4.2.1). As a consequence, the difference $A_\varepsilon - A_0$ is of size $O(1)$, without being necessarily better. Rather, following Remark 7, we compare B_ε with B_0.

$$\begin{aligned}
B_\varepsilon - B_0 &= \nabla_\varepsilon \times A_\varepsilon - {}^t({}^t\nabla_\perp, 0) \times A_0 \\
&= \mathrm{Op}\left(\frac{i}{\langle \xi_\varepsilon\rangle}\, \frac{\xi_\varepsilon}{|\xi_\varepsilon|} \times\right) \tilde A^* - \mathrm{Op}\big(i\; {}^t({}^t\xi_\perp, 0) \times A_0\big) \\
&= \mathrm{Op}\left(\frac{i}{\langle \xi_\varepsilon\rangle}\, \frac{\xi_\varepsilon}{|\xi_\varepsilon|} \times\right) \big(\tilde A^* - \psi_\varepsilon(D)(A^*_0 + \varepsilon A^*_1)\big) \\
&\quad + \mathrm{Op}\left(\frac{i\,\xi_\varepsilon}{\langle \xi_\varepsilon\rangle^2} - \frac{i}{\langle \xi_\perp\rangle^2}\, {}^t({}^t\xi_\perp, 0)\right) \times A^*_0 + \varepsilon\, \mathrm{Op}\left(\frac{i\,\xi_\varepsilon}{\langle \xi_\varepsilon\rangle^2}\right) \times A^*_1\,.
\end{aligned}$$

On the one hand, exploiting (5.11), we get that

$$\| \tilde A^* - \psi_\varepsilon(D)(A^*_0 + \varepsilon A^*_1) \|_{L^\infty([0,T];H^s(\Omega))} = O(\varepsilon)\,.$$

On the other hand, we have

$$\left| \frac{\xi_\varepsilon}{\langle \xi_\varepsilon\rangle^2} - \frac{{}^t({}^t\xi_\perp, 0)}{\langle \xi_\perp\rangle^2} \right| = \left| - \frac{\varepsilon^2\, \xi_3^2}{\langle \xi_\varepsilon\rangle^2\, \langle \xi_\perp\rangle^2}\, {}^t({}^t\xi_\perp, 0) + \frac{\varepsilon\, \xi_3}{\langle \xi_\varepsilon\rangle^2} \right| \leq 2\, \varepsilon\, |\xi_3|\,.$$

Since A^*_0 is in H^s with $s > 5/2$, $\partial_z A^*_0$ is in H^{s-1}. This implies that

$$\left\| \mathrm{Op}\left(\frac{i\,\xi_\varepsilon}{\langle \xi_\varepsilon\rangle^2} - \frac{i}{\langle \xi_\perp\rangle^2}\, {}^t({}^t\xi_\perp, 0)\right) \times A^*_0 \right\|_{L^\infty([0,T];H^{s-1}(\Omega))} = O(\varepsilon)\,.$$

By this way, we recover (1.11), which is quite meaningful because it shows that the magnetic field B_ε can indeed be compared to B_0 with a good $O(\varepsilon)$ degree of approximation.

Data availability

No data was used for the research described in the article.

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