Alfvén wave filamentation beyond the envelope formalism

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Abstract

Alfvén wave filamentation is shown to persist in the context of a long-wavelength asymptotics where the wave amplitude is larger than permitted by the envelope formalism. Furthermore, as predicted in this framework, reducing the dispersion transforms the collapsing structures from foci to gradient sheets where the wave amplitude remains moderate.

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1 Introduction

Alfvén waves, which are associated with transverse oscillations of magnetic field lines, play an important role in space plasmas which are usually permeated by an ambient magnetic field. When, within a magnetohydrodynamic description, the inertia of the ions is retained through the Hall term in a generalized Ohm's law, these waves are dispersive. Their dissipation is in contrast very weak when the plasma is assumed to be fully-ionized, in particular because the ambipolar diffusion due to the friction of the ions and the neutrals is negligible. In this context, the system is described by the so-called "Hall-MHD" equations which, in the case of a barotropic gas, read

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0 \tag{1}$$

$$\rho(\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) = -\frac{\beta}{\gamma}\nabla\rho^{\gamma} + (\nabla\times\mathbf{b})\times\mathbf{b} \tag{2}$$

$$\partial_t\mathbf{b} - \nabla\times(\mathbf{u}\times\mathbf{b}) = -\frac{1}{R_i}\nabla\times\Big(\frac{1}{\rho}(\nabla\times\mathbf{b})\times\mathbf{b}\Big) \tag{3}$$

$$\nabla\cdot\mathbf{b} = 0. \tag{4}$$

As usual, ρ is the density of the gas, $\mathbf{u}$ its velocity and $\mathbf{b}$ the magnetic field. The equations are written in nondimensional units with the Alfvén velocity $c_A = B_0/\sqrt{4\pi\rho_0}$

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taken as unity. In this formula, B_0 is the magnitude of the ambient magnetic field and ρ_0 the mean density of the plasma. The parameter R_i denotes the nondimensional ion-cyclotron frequency and γ the polytropic gas constant. The "beta of the plasma" is defined by $\beta = c_s^2/c_A^2$ where c_s is the sound velocity. The Hall-MHD equations are relevant for various media like the solar wind, the magnetospheres of planets, the solar corona, the interstellar medium, ...

The Hall-MHD equations admit exact solutions in the form of finite-amplitude monochromatic Alfvén waves that are circularly polarized and propagate parallel to the ambient field [1]. Assuming that this field points in the x-direction, the Alfvén-wave solutions have the form $b_y - i\sigma b_z = -\frac{\omega}{k}(u_y - i\sigma u_z) = Be^{i(kx-\omega t)}$ with $\sigma = \pm 1$, $u_x = 0$ and $\rho = 1$. Keeping $k > 0$, the frequency of a forward-propagating wave is given by the dispersion relation $\omega = \frac{\sigma}{2R_i}k^2 + k\sqrt{1 + \frac{k^2}{4R_i}}$. The choice $\sigma = +1$ or $\sigma = -1$ corresponds to a right-hand or left-hand circularly polarized wave respectively.

Parallel Alfvén waves can be subject to various instabilities. In one space dimension, one usually distinguishes between the decay and modulational instabilities [2]. The decay instability which involves two counter-propagating Alfvén waves and a magnetosonic wave affects "daughter waves" with wavenumbers K such as $K > k$, where k is the wavenumber of the carrier or pump wave. As a consequence, it can only be studied in the framework of the primitive Hall-MHD equations. The modulational instability affects daughter waves such that $K < k$. In the asymptotic regime where the unstable modes are such that $K \ll k$, the problem is amenable to a multiple-scale modulation analysis. When the amplitude of the carrier is small, the weakly nonlinear regime is described by the nonlinear Schrödinger equation for the Alfvén wave amplitude. As recently pointed out, retaining the coupling to the low-frequency magnetosonic waves allows one to predict at least qualitatively that, when β is located between the phase and group velocities of the carrier, the modulational instability occurs at wavenumbers comparable to k. The nonlinear regime is however poorly described, shocks developing on the primitive equations [3]. For waves with larger amplitude and long wavelength, the modulational instability can be described in the framework of a reductive perturbative expansion where the system is governed by the so-called derivative nonlinear Schrödinger (DNLS) equation (see [4] for a recent review). As long as no decay instability is present, a satisfactory description of the nonlinear regime is obtained, especially when the coupling with the magnetosonic wave is retained.

In higher dimensions, the complete theory of the linear instabilities for dispersive waves is rather cumbersome [5, 6]. Furthermore, a simple comparison of the growth rates of the various linearly unstable modes does not necessarily determine the dominant effect in the nonlinear regime, especially in the presence of singular phenomena. As a consequence, the nonlinear regime can only be investigated by direct simulations of the full Hall-MHD equations. Nevertheless, due to the many phenomena that are involved, it is of interest to first consider limiting cases where elementary processes can be isolated. Their detailed understanding seems to be a prerequisite to analyze a full simulation. We thus concentrate in the present paper on the nonlinear development of the filamentation instability in regimes amenable to asymptotic descriptions [7].

2 Envelope description of Alfvén wave filamentation

It is easily seen that whatever its polarization, a parallel (circularly polarized) Alfvén wave is unstable to transverse perturbations (filamentation instability). This instability is absolute (i.e. it develops in time) when $\beta > \frac{\omega}{k}$, while it is convective (i.e. it develops along the direction of propagation) when $\beta < \frac{\omega}{k}$. When, in the absolute regime, the modulation is purely transverse to the propagation or, in the convective regime, purely stationary, the nonlinear development of the filamentation instability [8] is governed by the two-dimensional nonlinear Schrödinger equation which for suitable initial conditions develops singularities in the form of amplitude blow-up at localized foci (see [9] for a recent review). A coupling to low-frequency magnetosonic waves comes into play when, in the absolute regime, the modulation is purely transverse or when, in the convective regime, the incident wavetrain has a finite duration. In this context, sharp magnetosonic fronts can develop which in some instances lead to the arrest of the collapse [10]. They nevertheless appear as an additional mechanism for small-scale formation which, when damping processes are retained, can provide an additional mechanism for the dissipation of the wave and the heating of the plasma.

It turns out that the limit of finite dispersion can be taken in the amplitude equations governing the Alfvén wave filamentation, but the consistency of the multiple-scale modulation requires that the wave amplitude be reduced at the same rate as the dispersion when this limit is taken. The dissipation resulting from wave collapse in this regime is estimated in [11]. If the dispersion is too weak to prevent resonant interactions with the carrier harmonics, a fully nonlinear regime can indeed develop. An intermediate regime can in fact be considered where the dispersion is sufficient to preserve the quasi-monochromatic character of the Alfvén wave but it is not strong enough to enforce circular polarization [12]. This is a regime where the parameter $\nu = \frac{l_d}{\lambda}\left(\frac{B}{B_0}\right)^{-2} = \frac{k}{2R_i}\left(\frac{B}{B_0}\right)^{-2}$ constructed with the dispersion length l_d, the carrier wavelength λ and its amplitude $\frac{B}{B_0}$ normalized to the ambient field, is large compared to unity but nevertheless finite. When a modulation analysis is performed in this context, the equations governing the components of the velocity and magnetic field in the two transverse directions are decoupled and the associated solvability conditions do not lead anymore to the usual scalar nonlinear Schrödinger equation but rather to a vector equation describing the coupling between the right and left polarized waves. The envelope equations for the vector amplitude $\mathbf{B} = (0, B_y, B_z)$, obtained in the limit of absolute purely transverse or convective stationary modulation, are formally similar up to the values of the coupling coefficients. Figure 1 and 2 reproduced from [12], display the result of the numerical integration of the envelope equation which in the

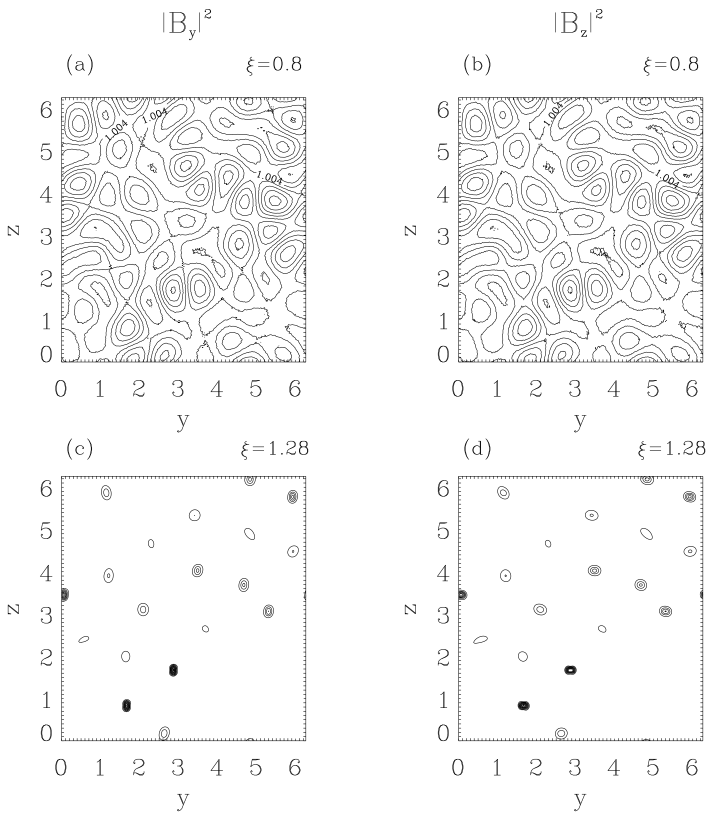

Figure 1: Contours of $|B_y|^2$ and $|B_z|^2$ at increasing propagation distances, for $\nu = 200$. Initial conditions consist in a circularly polarized wave perturbed by a noise.

stationary convective case reads

$$i\partial_\xi \mathbf{B} + \sigma\nu k\mathbf{B} + i\nu k(\mathbf{B}\times\mathbf{e}_1) + \frac{1}{2k(1-\beta)}\nabla(\nabla\cdot\mathbf{B})$$
$$+\frac{(-8\beta^2+5\beta+2)k}{4\beta(1-\beta)}|\mathbf{B}|^2\mathbf{B} + \frac{k}{4(1-\beta)}\mathbf{B}\times(\mathbf{B}\times\mathbf{B}^*) = 0. \qquad (5)$$

In this equation, the diffraction term is anisotropic and originates from the gradient of the total pressure in the primitive momentum equation. In the present small dispersion regime, the dispersion operator in the envelope equation acts on the fast rather than on the slow variables and reduces to the multiplicative factor $k\nu \propto k^2$. Here $\mathbf{e}_x$ denotes the unit vector in the direction of the ambient field and one easily checks that in the limit $\nu \to \infty$ the wave is circularly polarized with an amplitude governed by the scalar two-dimensional Schrödinger equation taken in the limit $k \to 0$. We note that the evolution of a plane wave $B_y = -iB_z = 1$ subject to to a weak random noise is strongly sensitive to the strength of the dispersion. In the strongly dispersive case $\nu = 200$ displayed in Fig. 1, the filamentation instability leads to a wave collapse with amplitude blowup at localized foci, as predicted by the scalar Schrödinger equation. When the dispersion

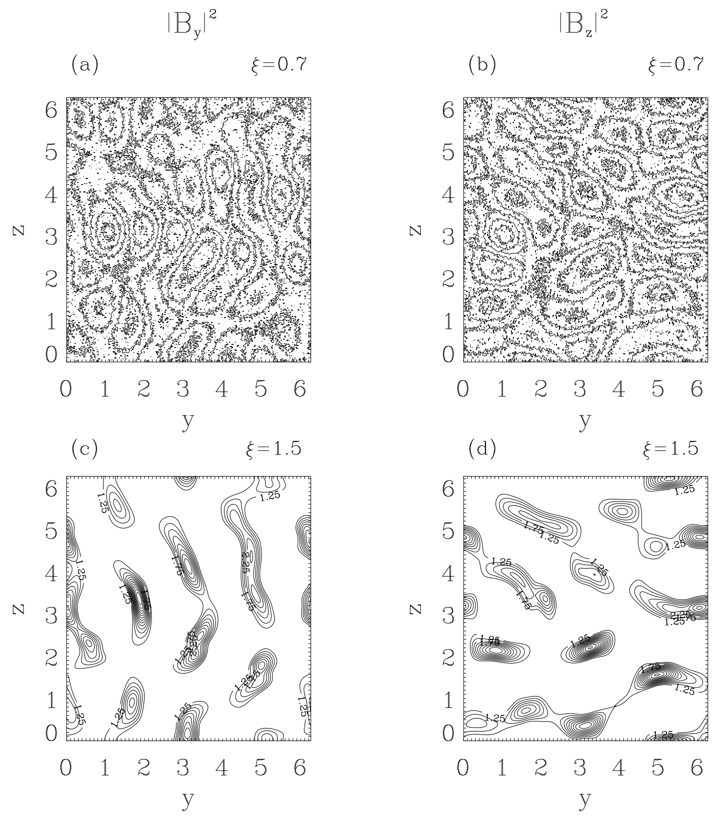

Figure 2: Same as Fig. 1 for $\nu = 10$.

is reduced to $\nu = 10$, small scales are formed but their nature is different. The wave amplitude remains moderate, while strong gradients concentrate in sheets with local maxima. As discussed in [12], the dissipation of the wave energy which takes place in the presence of a small-scale damping is sensitive to the strength of the dispersion. Although the small scale formation and thus the energy decay is postponed when the dispersion is weaker, the dissipation appears to be more homogeneous in the latter regime, and eventually leads to a globally more efficient energy decay.

3 Long-wavelength regime beyond the envelope dynamics

The Alfvén wave filamentation as addressed in the context of envelope equations is limited to waves whose amplitude are small enough for the nonlinearity to be weak compared with dispersion at the scale of the carrier. In particular, the multiple-scale asymptotics breaks down near collapse.

In order to study the Alfvén wave filamentation beyond the regime governed by the envelope formalism, it is convenient to consider a wave whose wavelength is large compared with the gyromagnetic radius of the ions. In this regime, a long wavelength reductive perturbative expansion can be used to single out Alfvén waves whose amplitude is such that dispersion and nonlinearity balance at the scale of the carrier.

As well known, the phase velocity of a parallel propagating Alfvén wave identifies in the non dispersive limit with that of the fast (when $\beta < 1$) or slow (when $\beta > 1$) magnetosonic wave. Excluding the case $\beta = 1$ which is a triple point where sonic and Alfvén waves are to be treated on equal footing [13], the reductive perturbative expansion assumes a scaling where the transverse components of the velocity and magnetic field scale like $\epsilon^{1/2}$, while the longitudinal components are of order ϵ. In one space dimension, the definition of the stretched variables $\xi = \epsilon(x - t)$ and $\tau = \epsilon^2 t$ then leads to the derivative nonlinear Schrödinger equation (DNLS) for the transverse components of the magnetic field $b = b_y + ib_z$. This equation is integrable by inverse scattering technique [14]. Slow transverse effects ($\eta = \epsilon^{3/2}y$, $\zeta = \epsilon^{3/2}z$) are considered in [15], but this derivation concerns localized solutions. The description of the transverse instability of Alfvén wave trains requires that longitudinal mean fields be retained. In the absence of transverse mean fields (which are not driven by the Alfvén waves), one obtains a system of equations of the form [16]

$$\partial_\tau \widetilde{b} + \partial_\xi \left(\frac{1}{2}(\widetilde{b}\widetilde{P} + (\bar{u}_x + \frac{1}{2}\bar{b}_x)\widetilde{b} \right) - \frac{1}{2}\partial_\perp \widetilde{P} + \frac{i}{2R_i}\partial_{\xi\xi}\widetilde{b} = 0 \tag{6}$$

$$\partial_\xi \widetilde{b}_x + \frac{1}{2}\left(\partial_\perp^* \widetilde{b} + \text{c.c.} \right) = 0 \tag{7}$$

$$\partial_\tau \bar{u}_x = \frac{1}{2}\left(\partial_\perp^* \langle \widetilde{b}\widetilde{P} \rangle + \text{c.c.} \right) \tag{8}$$

where tildes refer to fluctuating quantities and bars or $\langle \cdot \rangle$ to the average over the ξ-variable. Moreover, $\partial_\perp = \partial_\eta + i\partial_\zeta$. The fluctuations of magnetic pressure are given by $\widetilde{P} = \frac{1}{2(1-\beta)}(2\widetilde{b}_x + |\widetilde{b}|^2 - \langle |\widetilde{b}|^2 \rangle)$ and the mean induced longitudinal magnetic field by $\bar{b}_x = -\frac{\langle |\widetilde{b}|^2 \rangle}{2(\beta+1)}$.

These equations were integrated numerically with two initial conditions corresponding to a perturbed circularly polarized plane wave with a wavenumber 4 or 0.4 and an amplitude of 0.25 (in the DNLS variables) with, at large scale, a superimposed noise depending on the transverse variables. In this context, we choose to consider the absolute filamentation instability by taking $\beta = 3$, the convective regime which involves non trivial boundary conditions being much more delicate to reproduce numerically at the level of the DNLS or of the primitive MHD equations. The former initial condition corresponds to a regime where the dispersion dominates the nonlinearity by a factor $\nu = 32$ while for the latter this ratio is ten times smaller. Typical results are displayed in Figs. 3 and 4 for these two regimes respectively. In the case of the larger dispersion, a wave collapse similar to that predicted by the scalar NLS equation is observed. This collapse nevertheless saturates after the wave amplitude has been amplified by an order of magnitude. For the lower dispersion, a significantly different dynamics is observed. The wave amplitude evolves in a way qualitatively similar to that predicted by the

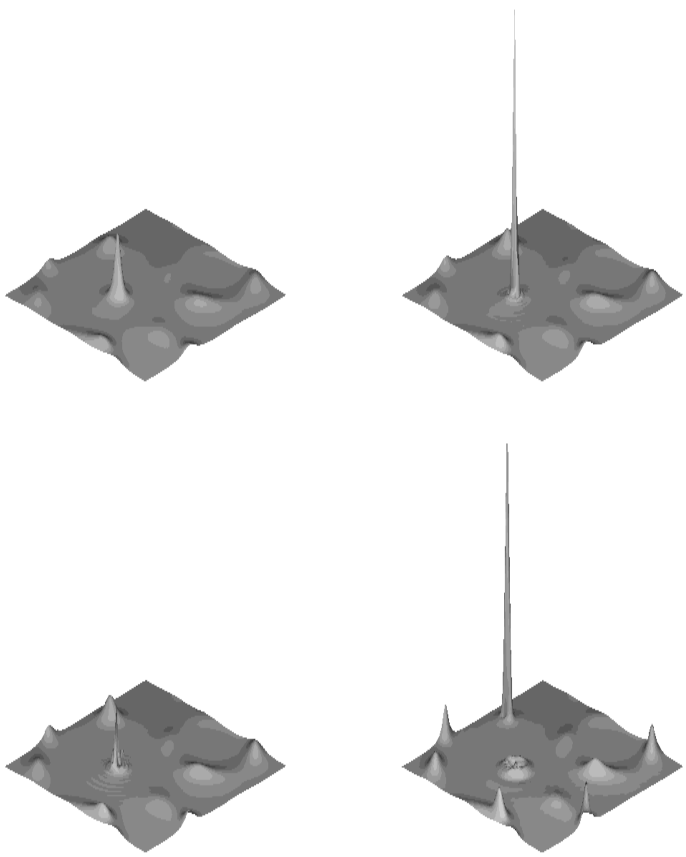

Figure 3: Intensity in a plane transverse to the propagation of an initially weakly perturbed circularly polarized Alfvén wave, at increasing times, for $\nu = 32$.

envelope equations in the weak dispersion limit. The amplitude of the wave remains moderate and the small scale formation is associated with the development of sheets of strong gradients. Although preliminary, these simulations give confidence in the envelope formalism for the description of Alfvén wave filamentation, even in regimes where the wave amplitude is not extremely small or the dispersion strongly dominant. The simulations displayed in Fig. 3 and 4 were performed with spectral methods at a resolution of 128^2 points in the transverse variables. We checked that retaining only one harmonic of the carrier in the longitudinal direction is sufficient to describe the collapse, which is consistent with the fact that at the level of the amplitude equations, the carrier interacts with the mean fields and not with its second harmonics [10, 12]. Even with this moderate resolution, the simulation of the wave collapse which, depending on $\nu = 32$ or $\nu = 3.2$, requires a few tens or a few hundreds of time units to develop, is very time-consuming because a time step $\delta\tau = 2\ 10^{-4}$ is needed to prevent small-scale numerical instabilities. In order to avoid significant truncation errors, we also included a weak dissipation that does not affect the level of saturation of the wave amplitude when the collapse is arrested.

It is of interest to notice that the long-wavelength reductive perturbative expansion can be generalized to include the coupling to magnetosonic waves whose effect on the Alfvén wave filamentation was discussed in [10] in the context of the envelope

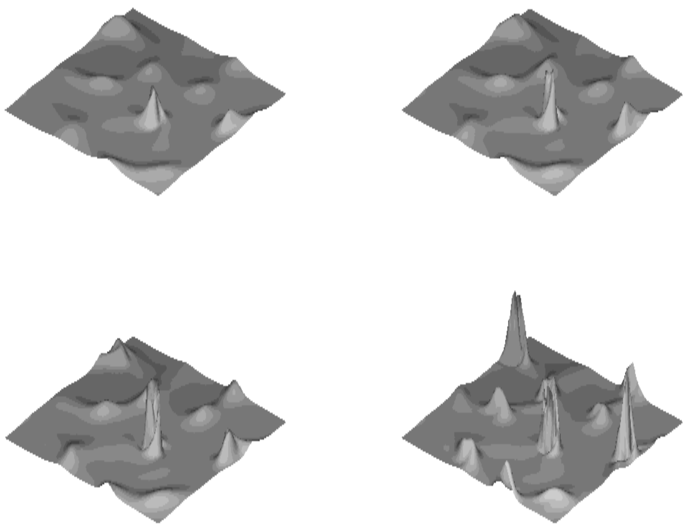

Figure 4: Same as Fig. 3 for $\nu = 3.2$.

formalism, and also to large-scale transverse velocity and magnetic field components which evolve on the same time scale as the nonlinear Alfvén waves and obey the "reduced MHD equations" that consist in the two-dimensional MHD equations for an incompressible flow [16]. To leading order, this transverse flow is not affected by the Alfvén waves but can in contrast drive the Alfvén waves and possibly provide a model where nonlinear Alfvén waves are fed by a transverse MHD turbulence.

4 Concluding remarks

In addition to their relevance in space plasmas, dispersive Alfvén waves can be viewed as an interesting paradigm of transition to turbulence in a dispersive medium. The amplitude formalism has shed light on the influence of the dispersion magnitude on the nature of Alfvén wave collapse. It is remarkable that the predictions of the envelope formalism concerning the modification of the resulting small-scale structures when the dispersion is reduced, are robust when the wave amplitude is increased and still hold in the context of the long wavelength regime described by a generalized DNLS equation. The DNLS description also displays the saturation of the amplitude blowup which occurs at large dispersion. The same formalism should also permit one to address the effect of the broadening of the Alfvén wave spectrum on the filamentation instability and its nonlinear development. Ultimately, it will be necessary to investigate the Alfvén wave filamentation by direct simulations of the primitive MHD equations which retain backward propagating waves that are excited by decay type instabilities, known to take place in strongly magnetized plasmas ($\beta < 1$).
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